Lower Semicontinuity of the Solution Set Mapping in Some Optimization Problems

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Zero Sum Games

An $n \times m$ matrix $P = (p_{ij})$, where $p_{ij} \in \mathbb{R}$ for all i, j, represents a two player, finite, zero-sum game: player one chooses a row i, player two a column j, and the entry p_{ij} of the matrix P so determined is the amount the second player pays to the first one.

$$\begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Zero Sum Games

$$\begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Suppose there exist a value v, row $\bar{\imath}$ and column $\bar{\jmath}$ such that $p_{\bar{\imath}j} \geq v$ for all j and $p_{i\bar{\jmath}} \leq v$ for all i: the first player is able to guarantee himself *at least* v, the second player can guarantee to pay *no more than* v. In particular (from the first inequality) $p_{\bar{\imath}\bar{\jmath}} \geq v$ and (from the second inequality) $p_{\bar{\imath}\bar{\jmath}} \leq v$: $p_{\bar{\imath}\bar{\jmath}} = v$ is the rational outcome of the game, with $(\bar{\imath}, \bar{\jmath})$ a pair of optimal strategies for the players.

From the second example: Need of mixed strategies.

Zero Sum Games

Theorem (von Neumann)

A two player, finite, zero sum game as described by a payoff matrix P has equilibrium in mixed strategies.

Finding optimal strategies

The first player must choose $S_n \ni \alpha = (\alpha_1, \dots, \alpha_n)$:

$$\alpha_1 p_{1j} + \dots + \alpha_n p_{nj} \geq v, \quad 1 \leq j \leq m,$$

v as large as possible.
Suppose (wlog) $p_{ij} > 0$, thus $v > 0$.
With the change of variable $x_i = \frac{\alpha_i}{v}$:
$\sum_{i=1}^m \alpha_i = 1$ becomes $\sum_{i=1}^m x_i = \frac{1}{v}$:
maximizing v is equivalent to minimizing $\sum_{i=1}^m x_i$.
The first player problem:

$$\begin{cases} \inf \langle 1_n, x \rangle \text{ such that} \\ x \geq 0, P^T x \geq 1_m \end{cases} .$$

Second player

$$\begin{cases} \sup \langle 1_m, y \rangle \text{ such that} \\ y \geq 0, Py \leq 1_n \end{cases} .$$

Finite General Games

A bimatrix $(A, B) = (a_{ij}, b_{ij})$, $i = 1, \dots, n$, $j = 1, \dots, m$ is a finite game in strategic form (a_{ij}, b_{ij} utilities of the players). (In the zero sum case $b_{ij} = -a_{ij}$). Let $I = \{1, \dots, n\}$, $J = \{1, \dots, m\}$ and $X = I \times J$. Let Δ_k denote the standard simplex in $\mathbb{R}^k$. A Nash equilibrium is a pair $(\bar{p}, \bar{q})$, $(\bar{p} \in \Delta_n)$, $(\bar{q} \in \Delta_m)$ such that

$$\sum_{i,j} \bar{p}_i \bar{q}_j a_{ij} \geq \sum_{i,j} p_i \bar{q}_j a_{ij}$$

$$\sum_{i,j} \bar{p}_i \bar{q}_j b_{ij} \geq \sum_{i,j} \bar{p}_i q_j b_{ij}$$

for all $p \in \Delta_n, q \in \Delta_m$. If both p, q are extreme points of the simplexes: pure Nash equilibria, if inequalities are strict for $p \neq \bar{p}$ and $q \neq \bar{q}$: strict Nash equilibria.

Finding Nash Equilibria

Notation: $f(p,q) = \sum_{i,j} p_i q_j a_{ij}$.

$$BR_1 : Y \to X : BR_1(y) = \mathrm{Max}\,\{f(\cdot, y)\}$$

$$BR_2 : X \to Y : BR_2(x) = \mathrm{Max}\,\{g(x, \cdot)\},$$

$$BR : X \times Y \to X \times Y : \quad BR(x,y) = (BR_1(y), BR_2(x)).$$

A Nash equilibrium for a game is a fixed point for BR (Best Reaction).

Correlated Equilibria

A correlated equilibrium is a probability distribution $p = (p_{ij})$ on X such that, for all $\bar{\imath} \in I$,

$$\sum_{j=1}^{m} p_{\bar{\imath}j} a_{\bar{\imath}j} \geq \sum_{j=1}^{m} p_{\bar{\imath}j} a_{ij} \qquad \forall i \in I,$$

and such that, for all $\bar{j} \in J$

$$\sum_{i=1}^{n} p_{i\bar{\jmath}} b_{i\bar{\jmath}} \geq \sum_{i=1}^{n} p_{i\bar{\jmath}} b_{ij} \qquad \forall j \in J.$$

Example 1

Example

$$\begin{pmatrix} (6,6) & (2,7) \\ (7,2) & (0,0) \end{pmatrix}.$$

Nash outcomes: $(2,7)$, $(7,2)$ (pure), also a mixed providing $\frac{14}{3}$ to both.

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}.$$

is a "nice" correlated equilibrium.

Example 2

Example

$$\begin{pmatrix} (5,5) & (0,5) \\ (5,0) & (1,1) \end{pmatrix}.$$

$(5,5)$ is a "fragile" (yet interesting) Nash equilibrium.

Aim of the paper(s)

1. to generalize former results of lower stability in linear inequality systems in two ways:
   - by allowing also restricted perturbations
   - by restricting the solution map to its effective domain
2. to apply the machinery to equilibria in games

Some notation

Linear inequality systems in $\mathbb{R}^n$, with arbitrary index set T, i.e. systems of the form

$$\sigma = \{a_t' x \leq b_t,\ t \in T\},$$

$a : T \to \mathbb{R}^n$, $b : T \to \mathbb{R}$.
We shall identify σ with the data (a, b), so that the *parametric space* is $\Theta = (\mathbb{R}^{n+1})^T$.
The *solution set mapping* (or *feasible set mapping*) is $\mathcal{F} : \Theta \rightrightarrows \mathbb{R}^n$ such that $\mathcal{F}(\sigma) = \{x \in \mathbb{R}^n : a_t' x \leq b_t,\ t \in T\}$, with domain
$\text{dom}\ \mathcal{F} = \{\sigma \in \Theta : \mathcal{F}(\sigma) \neq \emptyset\}$.
Solution set mapping relative to its domain the restriction of $\mathcal{F}$ to dom$\mathcal{F}$, denoted by $\mathcal{F}^R$.
When one parameter is fixed we write a subscript, so when a is fixed $\mathcal{F}_a$, Θ_a, when b is fixed $\mathcal{F}_b$, Θ_b.

Definitions

Definition

$\widehat{x}$ *is a Slater point for* σ *if*

$$a_t'\widehat{x} < b_t, \ \forall t \in T;$$

$\widehat{x}$ *is a strong Slater point (SS) for* σ *if there exists* $\rho > 0$ *such that*

$$a_t'\widehat{x} + \rho \leq b_t, \ \forall t \in T.$$

σ *satisfies the (strong) Slater condition if there is a (strong) Slater point for* σ

Definition

A consistent system σ *is continuous whenever* T *is a compact Hausdorff topological space and* $a : T \to \mathbb{R}^n$ *and* $b : T \to \mathbb{R}$ *are continuous.*

Main Reference Theorem

Theorem

The following are equivalent:

1. σ *has a strong Slater point*
2. $\mathcal{F}$ *is lower semicontinuous at* σ
3. $\sigma \in \mathrm{int dom} \mathcal{F}$
4. $\mathcal{F}$ *is dimensionally stable at* σ.

First Results: topological structure of the domains of $\mathcal{F}$, $\mathcal{F}_a$, $\mathcal{F}_b$

Proposition

- *dom* $\mathcal{F}$ *is neither open nor closed*
- $\mathcal{F}_a$, $\mathcal{F}_b$ *can be open, closed, or both*

Complete characterizations in the second case:

Topological Properties

Perturbing the RHS

Proposition

The following statements hold true:

- $\mathrm{dom}\mathcal{F}_a = \Theta_a$ *if and only if* T *is finite and* $0_n \notin \mathrm{conv}\{a_t, t \in T\}$
- $\mathrm{dom}\ \mathcal{F}_a$ *is open in* Θ_a *if and only if* $0_n \notin \mathrm{clconv}\{a_t, t \in T\}$
- *If* T *is finite, then* $\mathrm{dom}\mathcal{F}_a$ *is closed in* Θ_a *for any* $a \in (\mathbb{R}^n)^T$.

Perturbing the LHS

Proposition

The following statements hold true:

- $\mathrm{dom}\mathcal{F}_b = \Theta_b$ *if and only if* $b_t \geq 0$ *for all* $t \in T$
- $\mathrm{dom}\mathcal{F}_b$ *is an open proper subset of* Θ_b *if and only if* $\sup_{t \in T} b_t < 0$
- $\mathrm{dom}\mathcal{F}_b$ *is closed in* Θ_b *if and only if* $b_t \geq 0$ *for all* $t \in T$.

Perturbing All data

Proposition

Given $\sigma \in \mathrm{dom}\mathcal{F}$, the following statements are true:
(i) If either σ satisfies SSC or $\mathcal{F}(\sigma)$ is a singleton set, then $\mathcal{F}^R$ is lsc at σ.
(ii) If $\mathcal{F}^R$ is lsc at σ and $\mathcal{F}(\sigma)$ is not a singleton set, then σ satisfies SSC.

Proposition

Let $\sigma \in \mathrm{dom}\mathcal{F}$ be a continuous system without trivial inequalities. TFAE:

- *$\mathcal{F}^R$ is dimensionally stable at σ*
- *the SSC holds*
- $\dim \mathcal{F}(\sigma) = n$.

Whence $\mathcal{F}^R$ is lsc at σ if and only if $\dim \mathcal{F}(\sigma) \in \{0, n\}$.

Perturbing the RHS

Proposition

Let $\sigma = \{a_t' x \leq b_t, t \in T\} \in \mathrm{dom}\mathcal{F}_a$. If either $0_{n+1} \notin \mathrm{clconv}\{(a_t, b_t), t \in T \setminus T_0\}$ or $\mathcal{F}(\sigma)$ is a singleton set, then $\mathcal{F}_a^R$ is lsc at σ.

Proposition

Let $\sigma \in \mathrm{dom}\mathcal{F}_a$ be a continuous system without trivial inequalities. TFAE:

- $\mathcal{F}_a^R$ is *dimensionally stable* at σ
- *SSC holds*
- $\dim \mathcal{F}(\sigma) = n$.

As a result, if $\dim \mathcal{F}(\sigma) \in \{0, n\}$, then $\mathcal{F}_a^R$ is lsc at σ.

Proposition

If T is finite, then $\mathcal{F}_a^R$ is lsc at any $\sigma \in \mathrm{dom}\mathcal{F}_a$.

Perturbing the LHS

Proposition

Let $\sigma \in \mathrm{dom}\mathcal{F}_b$ be a continuous system without trivial inequalities such that $0_n \notin \mathcal{F}(\sigma)$. TFAE:

- $\mathcal{F}_b^R$ *is dimensionally stable at* σ
- *the SSC holds*
- $\dim \mathcal{F}(\sigma) = n$.

Moreover, any of these properties implies that $\mathcal{F}_b^R$ *is lsc at* σ.

Proposition

Assume that T is finite and that σ contains no trivial inequality. Then $\mathcal{F}_b^R$ is lsc at $\sigma \in \mathrm{dom}\mathcal{F}$ if and only if one of the following alternatives holds:

- $\dim \mathcal{F}(\sigma) = n$
- $\dim \mathcal{F}(\sigma) = 0$
- $\mathcal{F}(\sigma)$ *is a non-singleton set contained in some open ray.*

Correlated Equilibria:the Two by Two Case

Proposition (the two by two case)

In a two by two bimatrix game the following happens:

- *If there are no dominated strategies, then all correlated equilibria are lower stable*
- *in presence of dominated strategies, only pure strict Nash equilibria are lower stable*

Correlated Equilibria:the Zero Sum Case

Proposition

A correlated equilibrium of a zero-sum game is stable within the space of zero-sum games if and only if it is the unique correlated equilibrium of the game.

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