A glance at Game Theory

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Summary

Basic assumptions

Player (Agent) is:

Selfish

Rational

Selfish

Every agent tries to get the best for himself, without taking care of the satisfaction of other agents

Rational

Premise Each player has some alternatives to choose from, and every combination of one choice for each player produces a possible outcome of the game.

From this

- Players are able to order the outcomes of the game, according to their preferences
- Players are consistent in their expectations
- Players are able to make a *unlimited* analysis of the game
- and much more...

Ordering the outcomes

The player is able to provide a complete preorder on the outcomes

In a general setting this means to have a utility function defined on the outcomes of the game

However Transitivity. . .

Consistent expectations. Two famous lotteries

Alternative A

gain	probability
2500	33%
2400	66%
0	1%

Alternative B

gain	probability
2500	0%
2400	100%
0	0%

Alternative A

gain	probability
2500	33%
0	67%

Alternative B

gain	probability
2400	34%
0	66%

Observe:

$$2500 \times 0.33 + 2400 \times 0.66 + 0 \times 0.01 > 2400.$$

However in a sample of 72 people exposed to the first lottery, 82% of them decided to play the Lottery B.
Introducing a utility function, the result can be read as:

$$\frac{34}{100}u(2400) > \frac{33}{100}u(2500).$$

Problem: 83% of the same people interviewed selected Lottery A in the second case!

(A first Nobel Prize: Allais)

Unlimited Analysis: the beauty contest

Each of you writes, on a sheet of paper, a positive integer not greater than 100.
The winners are those writing the integer closest to the $2/3$ of the average of the audience

(Keynes)
It is not a case of choosing those [faces] which, to the best of ones judgement, are really the prettiest, nor even those which average opinion genuinely thinks the prettiest. We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practice the fourth, fifth and higher degrees.

Much More

A first basic assumption:

A player does not choose an action X, if it is available to him an action Z allowing to him to get more, no matter which choice will make the other players

Terminology: X is dominated by Z.

A first example

Game in bimatrix form:

$$\begin{pmatrix} (5,5) & (6,1) \\ (1,6) & (4,4) \end{pmatrix}$$

$$\begin{pmatrix} (5,) & (6,) \\ (1,) & (4,) \end{pmatrix}$$

$$\begin{pmatrix} (,5) & (,1) \\ (,) & (,) \end{pmatrix}$$

$$\begin{pmatrix} (5,5) & (,) \\ (,) & (,) \end{pmatrix}$$

A first unexpected consequence

First situation

$$\begin{pmatrix} (10, 10) & (3, 15) \\ (15, 3) & (5, 5) \end{pmatrix},$$

Second situation

$$\begin{pmatrix} (8, 8) & (2, 7) \\ (7, 2) & (0, 0) \end{pmatrix}.$$

Which game is more convenient to play?

A second unexpected consequence

Restricted strategies

$$\begin{pmatrix} (10,10) & (3,5) \\ (5,3) & (1,1) \end{pmatrix}$$

Extended strategies

$$\begin{pmatrix} (1,1) & (11,0) & (4,0) \\ (0,11) & (10,10) & (3,5) \\ (0,4) & (5,3) & (1,1) \end{pmatrix}$$

Which game is more convenient to play?

Domination vs strong domination

A voting situation

$$\begin{aligned} A &\succ B \succ C, \\ B &\succ C \succ A, \\ C &\succ A \succ B. \end{aligned}$$

In case of three different votes the outcome is what the first voted

1. The first votes her dominant choice
2. the second and third eliminate the worst outcome

Reduction

$$\begin{pmatrix} A & A \\ C & A \end{pmatrix}$$

the second is the row player

Outcome: C

The most famous example of the whole theory

$$\begin{pmatrix} (10, 10) & (0, 11) \\ (11, 0) & (1, 1) \end{pmatrix}$$

The prisoner dilemma

An example of perfect information

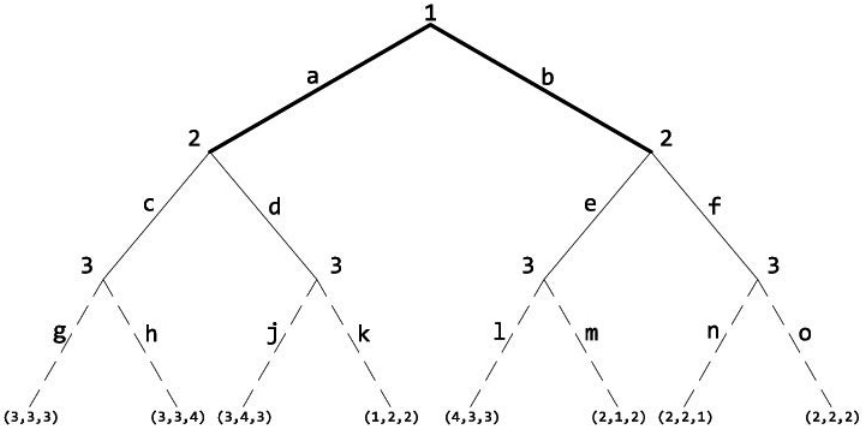

Figure: The game of the three politicians

An example of perfect information

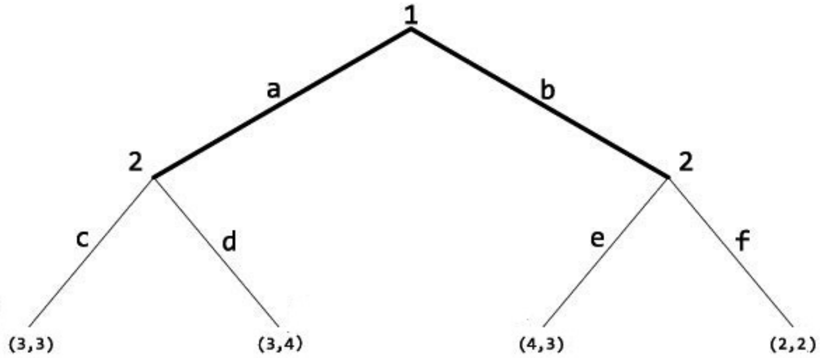

Figure: The game of the three politicians:continued

An example of perfect information

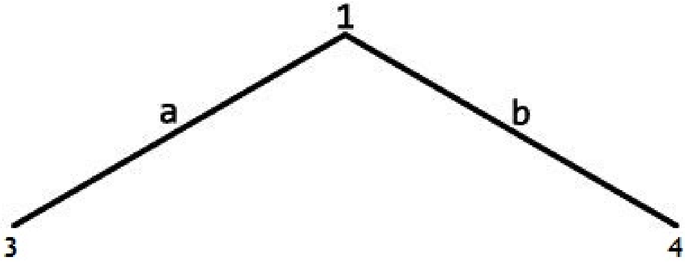

Figure: The game of the three politicians:conclusion

Chess

The above method, called Backward Induction, applies to all games where everything is known to players, moves are in sequence, choices are in finite number, length is finite

The so called Zermelo Theorem

In the game of chess, it holds one and only one of the following alternatives:

1. The white has a strategy making him win, no matter what strategy will use the black
2. The black has a strategy making her win, no matter what strategy will use the white
3. The white has a strategy making him to get at least the draw, no matter what strategy will use the black, and the same holds for the black

Visualizing Chess theorem

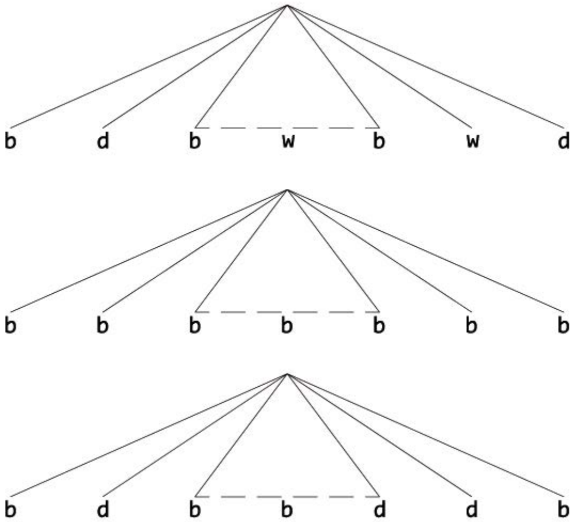

Figure: The three alternatives

Another interesting game: the Chomp game

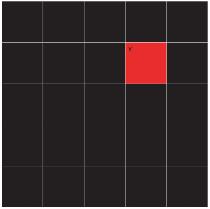

Figure: The first player removes the red square

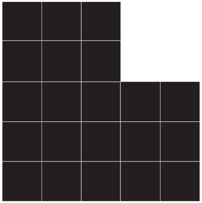

Figure: Now it is second player's turn

The winner in the chomp game

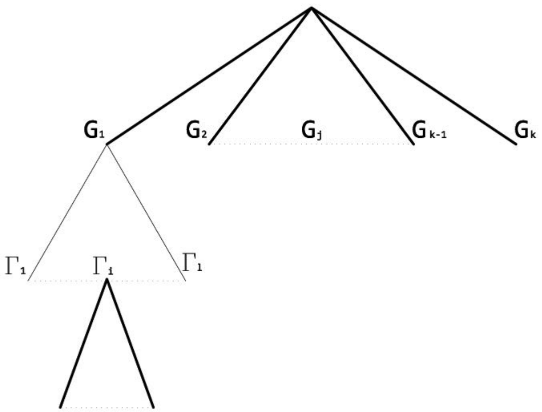

Figure: Proving that the first player has a winning strategy

Comments

Backward induction is a brilliant idea, but how applicable?

The exact number of nodes of the tree when limited to length eleven (1 correspond to length zero and represents the root of the tree):

Chess [1, 20, 400, 8902, 197281, 4865609, 119060324, 3195901860, 84998978956, 2439530234167, 69352859712417, 2097651003696806]

Checkers [1, 7, 49, 302, 1469, 7361, 36768, 179740, 845931, 3963680, 18391564, 85242128]

Different types of solutions

Very weak solution

As in the chess case: the game is determined, but the outcome is not known

Weak solution

As in the chomp case, the winner is known, but no strategy is available. The same is with checkers.

Strong solution

A winning strategy can be displayed.

The Nim game

A certain number of piles of cards. Each player can take as many cards as she wants, but only from one pile. Who clears the table wins

(A class of) Combinatorial game(s): *two players,consecutive moves, no chance, no tie*.

F.i. with two piles the first wins unless the two piles have the same number of cards.

An abelian group

For n, m non negative integers:

1. write n, m in binary base: $n = (n_k \ldots n_0)_2$, $m = (m_k \ldots m_0)_2$
2. form the binary number $(n_k + m_k)_2 \ldots (n_0 + m_0)_2$
3. define the operation $n +_N m = z$, where $(z)_2 = ((n_k + m_k)_2 \ldots (n_0 + m_0)_2)$.

Theorem

The set of the nonnegative integers with $+_N$ in an abelian group.

Proof.

The identity element is 0, the inverse of n is n itself. Associativity and commutativity of $+_N$ are obvious. □

The Bouton theorem

Theorem

A $(n_1, n_2, \ldots, n_k)$ position in the Nim game is a winning position for the second player if and only if $n_1 +_N n_2 +_N \ldots +_N n_k = 0$.

Idea of the proof

- Prove that every move from the situation

$$x_1 +_N x_2 +_N \ldots +_N x_k = 0$$

  brings to a situation

$$y_1 +_N y_2 +_N \ldots +_N y_k \neq 0$$

- Prove that there is a move from the situation

$$y_1 +_N y_2 +_N \ldots +_N y_k \neq 0$$

  bringing to a situation

$$x_1 +_N x_2 +_N \ldots +_N x_k = 0$$

A proof for combinatorial games

Express, in terms of consecutive moves, when **the first player has a winning strategy**:

At the root of the game there is, for the first player, a move such that, for every move of the second player at the second step, there is, at the next step, a move for the first player such that, for every move of the second player at the next step . . . the first wins.

Writing the negation of the above statement:

For every move of the first player at the root of the game, there is a move for the second player at the second stage such that, for every move of the first player at the next step, there is a move for the second player at the next step such that . . . , the first player does not win

If the first player does not win, then it is the second to win, since a tie is not allowed.

But the second statement means exactly that:

There is a winning strategy for the second player

The game

Players must consecutively say a digit, either 0 or 1. If at the n-th stage the number d_n is said, then the number

$$n = \sum_{n=1}^{\infty} \frac{d_n}{2^n}$$

is formed. A subset A of $[0,1]$ is given. The first wins if $n \in A$.

Examples $A = [\frac{1}{2}, 1]$, $A = [0, \frac{1}{4}]$.

Question Is the game determined for every A?

Answer Maybe Maybe not ...

Further developments

- Zero sum games
- The cooperative theory
- The non cooperative Nash model
- Repeated games
- Incomplete information
- Mechanism design
- And much more
- Applications