Probabilistic values and semivalues

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Summary

TU Games

We are given a *finite* set N, of cardinality n.

Definition

A TU-game on N is a function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$.$\mathcal{G}$ is the set of all games (with N fixed).

Remark

$\mathcal{G} \approx \mathbb{R}^{2^{n-1}}$. *A base for the space: the collection of the unanimity games.* The *unanimity game* (N, u_R), $\emptyset \neq R \subseteq N$, is the game described by

$$u_R(T) = \begin{cases} 1 & \text{if } R \subseteq T \\ 0 & \text{otherwise} \end{cases}.$$

Another base is the collection of games e_R:

$$e_R(T) = \begin{cases} 1 & \text{if } T = R \\ 0 & \text{otherwise} \end{cases}.$$

Power indices

Definition

A power index on $\mathcal{G}$ is a function $f : \mathcal{G} \to \mathbb{R}^N$.

The Shapley and Banzhaf indices are power indices.

Definition

A probabilistic index on $\mathcal{G}$ is a power index π of the form:

$$\pi_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} p_i(S) m_i(S)$$

where $m_i(S) = v(S \cup \{i\}) - v(S)$ is the marginal contribution of i to $S \cup \{i\}$ and the coefficients $p_i(S)$ are non negative numbers fulfilling the condition $\sum_{S \in 2^{N \setminus \{i\}}} p_i(S) = 1$.

The Shapley and Banzhaf indices are probabilistic indices.

Semivalues

Definition

A semivalue is a probabilistic index such that $p_i(S) = p(|S|)$. If moreover $p(|S|) > 0$ for $|S| = 1, \ldots, n-1$, then the semivalue is called regular

The Shapley and Banzhaf indices are regular semivalues.

Notation $p_s = p(|S|)$.
$p_s = \frac{1}{n\binom{n-1}{s}}$ for Shapley, $p_s = \frac{1}{2^{n-1}}$ for Banzhaf,$p(s) = p^s(1-p)^{n-s-1}$, for $0 < p < 1$ defines the p-binomial semivalues The set of semivalues is the simplex made

by vectors $x = (x_0, \ldots, x_k, \ldots, x_{n-1})$:

$$\sum_{k=0}^{n-1} x_k \binom{n-1}{k} = 1$$

Properties of semivalues

Property

The power index f has the dummy player (DP) property, if for each player $i \in N$ such that $v(A \cup \{i\}) = v(A) + v(\{i\})$ for all $A \subset N \setminus \{i\}$, then

$$f_i(v) = v(\{i\}).$$

Property

Let $\pi : N \to N$ be a permutation of N. Given the game v, denote by $\pi^* v$ the following game: $(\pi^* v)(A) = v(\pi(A))$, and by $\pi^*(x) = (x_{\pi(1)}, \ldots, x_{\pi(n)})$. The power index f has the symmetry (S) property if, for each permutation π on N, $f(\pi^* v) = \pi^*(f(v))$.

Property

The power index f has the linearity (L) property if $f : \mathcal{G} \to \mathbb{R}^N$ is a linear functional.

The semivalues enjoy the (DP), (S) and (L) properties.

Properties of probabilistic indices

Theorem

A power index f is probabilistic if and only if it fulfills the (DP) and (L) properties, and the coefficients $f_i(e_{S\cup\{i\}})$ are non negative.

Semivalues on unanimity games

Given the unanimity game:

$$u_R(T) = \begin{cases} 1 & \text{if } R \subseteq T \\ 0 & \text{otherwise} \end{cases}.$$

An immediate calculation: Shapley value assigns

- 0 to the players not in R
- $\frac{1}{r}$ to the players in R

An easy calculation: Banzhaf assigns

- 0 to the players not in R
- $\frac{1}{2^{r-1}}$ to the players in R

In general not easy for binomial

Defining semivalues through unanimity games

Definition

Let $a \in \mathbb{R}$, $a > 0$. We shall denote by σ^a, and call a index, the solution on $\mathcal{G}$, defined on the unanimity game u_R as

$$\sigma_i^a(u_R) = \begin{cases} \frac{1}{r^a} & \text{if } i \in R \\ 0 & \text{otherwise} \end{cases},$$

and extended by linearity on $v \in \mathcal{G}$.

Main Theorem

Theorem

The a-value σ^a is a regular semivalue for all $a > 0$.
The 2-value *fulfills:*

$$\sigma_i^2(v) = \sum_{S \subseteq 2^{N \setminus \{i\}}} \left(\frac{s!(n-1-s)!}{n!} \sum_{k=s+1}^{n} \frac{1}{k} \right) m_i(S).$$

Main steps for the proof

The proof is based on the following steps.

- The characterization of the probabilistic coefficients $p_i(S)$ given by Weber:
$$p_i(S) = \sigma_i^a(e_{S\cup\{i\}})$$
- since the value σ^a is defined on the base of the unanimity games, we need to find a formula of change of base, passing from unanimity games to canonical games:

Proposition

Let e_T, $T \subseteq N$, be the family of games associated to the canonical base in $\mathbb{R}^{2^n-1}$ and let u_A, $A \subseteq N$, be the family of the unanimity games. Then the following formula holds:

$$e_T = \sum_{k=0}^{n-t}(-1)^k \sum_{A:a=k,A\cap T=\emptyset} u_{A\cup T}.$$

Main steps for the proof,continued

- Theorem There exists one and only one index ϕ fulfilling the (DP), (L) and (S), and assigning a_s to all non null players in the unanimity game u_S, for all coalitions S such that $|S| = s$, where $a_1 = 1$ and $a_s > 0$ for $s = 2, \dots, n$. Moreover ϕ fulfills the formula:

$$\phi_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} \left(\sum_{k=0}^{n-s-1} \binom{n-s-1}{k} (-1)^k a_{s+k+1} \right) m_i(S).$$

- Its Corollary: Suppose, for each $s = 1, \dots, n$, positive numbers a_s are given and suppose ϕ is a value fulfilling the null player, linearity and symmetry axioms, and assigning a_t to all non null players in the unanimity game u_T, for all coalitions T such that $|T| = t$. Then, for a player i and for a coalition S such that $i \notin S$, it holds:

$$\phi_i(e_{S \cup \{i\}}) = \sum_{k=0}^{n-s-1} \binom{n-s-1}{k} (-1)^k a_{s+1+k}.$$

Main steps for the proof,conclusion

Finally prove (This is the hard part indeed!) that in the above formula when $a_s = \frac{1}{s^a}$ then the coefficients are positive, and sum up to one (Less hard)

I.e. prove that

1. the coefficients:

$$\sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} \frac{1}{(s+k+1)^a}$$

   are positive;

2. their sum verifies

$$\sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} \frac{1}{(s+k+1)^a} = 1.$$

Corollary and generalization

Corollary The family of the weighting coefficients of the values σ^a, $a \in \mathbb{R}_+$, is an open curve in the simplex of the regular semivalues, containing the Shapley value. The addition of the Banzhaf value provides a one-point compactification of the curve.

To generalize:

Find conditions on the coefficients a_t, $t = 1, \ldots, n$, to guarantee the following two facts:

1. the coefficients:
$$\sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} a_{s+k+1}$$
are non negative;
2. their sum verifies
$$\sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} a_{s+k+1} = 1$$

I know a very nice answer, but not published yet!

Extensions of total preorders on the power set of a set

Why do we need so many semivalues? A well studied problem in literature tries to find coherent extensions of a preorder on a finite set of objects, to its power set

Well known example (RESP condition):
Given a total preorder $\succcurlyeq$ on N, a RESP extension $\sqsupseteq$ on 2^N is such that for all $i, j \in N$ and all $S \in 2^N$, $i, j \notin S$ then

$$i \succcurlyeq j \Rightarrow S \cup \{i\} \sqsupseteq S \cup \{j\}.$$

All extensions present in literature try to avoid interactions

between object, but this is a severe restriction.

Thus it is useful to find nice extensions allowing however some interaction

A hopefully! interesting idea

- A normalized utility function v representing the total preorder $\succcurlyeq$ on N is a TU game;
- An extension $\sqsupseteq$ on 2^N should have the property that the Shapley value calculated via v respects the ranking of the objects
- However this must be independent from the function v chosen to represent $\succcurlyeq$
- And why should we use (only) the Shapley value?

I have characterization on preorders $\sqsupseteq$ on 2^N enjoying this ordinality property both for a fixed semivalue and for the whole family of semivalues.

Using all probabilistic indices in this case is not interesting since being ordinal for every probabilistic index implies RESP

The end

Homage to Shapley

Figure: The airport game

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