Roberto Lucchetti*

A Primer in Game Theory

With Solved Exercises†

† The part of the Exercises is developed by Michela Chessa, Nash and von Neumann are drawn by Géraldine De Alessandris, Figures are due to Laura Stirnimann, Michela Chessa, Alberto Lucchetti. Thanks to them for their precious work

* Department of Mathematics, Politecnico di Milano, Via Bonardi 9, 20133 Milano, Italy, roberto.lucchetti@polimi.it

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Game theory and optimization

1.1 Introduction

1.1.1 What is Game Theory?

A game is an efficient model of interactions between agents, for the following basic reason: the players follow fixed rules, have interests on all possible final outcomes of the game, and the final result for them does not depend only from the choices they individually make, but also from the choices of other agents. Thus the focus is actually on the fact that in a game there are several agents *interacting*. In fact, more recently this theory took the name of *Interactive Decision Theory*. It is related to classical decision theory, but it takes into account the presence of *more than one* agent taking decisions. As we shall constantly see, this radically changes the background and sometimes even the intuition behind classical decision theory. So, in few words, game theory is the study of taking optimal decisions in presence of multiple players (agents).

1.1.2 What is Game Theory for?

Thus a game is a simplified, yet very efficient, model of real life every day situations. Though the first, and probably more intuitive, applications of the theory were in an economical setting, theoretical models and tools of this theory nowadays are spread on various disciplines. To quote some of them, we can start from psychology: a more modern approach than classical psychanalysis takes into account that the human being is mainly an interactive agent. So to speak, we play everyday with our professors/students, with our parents/children, with our lover, when bargaining with somebody. Also the Law and the Social Sciences are obviously interested in Game Theory, since the *rules* play a crucial role in inducing the behaviour of the agents. Not many years after the first systematic studies in Game Theory, interesting applications appeared to animals, starting with the analysis of competing species. It is much more recent and probably a little surprising to know that recent applications of the theory deal with genes in microbiology, or computers in telecommunication problems. In some sense, today many scholars do believe that these will be the more interesting applications in the future: for reasons that we shall constantly see later, humans in some sense are *not* so close to the rational player imagined by the theory, while animals and computers "act" in a more rational way than human beings, clearly in an unconscious yet efficient manner.

1.1.3 A few lines about the history of Game Theory

Mathematical game theory was essentially born in the last century, even if it is possible (as always) to find contributions in more ancient times. The first main result is traditionally considered an article by the mathematician Zermelo, stating some facts about the game of chess. Curiously, the result which nowadays is called the Zermelo theorem is not due to Zermelo and is not present in any form in that paper. The subsequent very important contributions are due to the mathematician von Neumann, and deal with the two players, finite, zero sum games, which are the games where the interests of the two players are strictly opposite. These results go back to the thirties and find their highest expression in the minimax theorem. The games where the interests of the players are strictly opposite do not represent well all real situations. It is easy to produce examples where the players can have partial common interests.[1] Thus it is important to extend the analysis to more complicated games. This has been done with essentially two different approaches. The first one has its foundations in the von Neumann-Morgenstern book *Theory of games and of economic behavior* (1944), and it is based on the so called cooperative theory, while few years later J.F. Nash Jr. proposed a noncooperative model, extending and formalizing previous ideas by Cournot, and leading to the central concept of Nash equilibrium, which was worth for him the Nobel Prize in Economics, in 1994. I stop here with quoting the history of early Game Theory, and I suggest the reader to take a look at the web page http://www.econ.canterbury.ac.nz/personal_pages/paul_walker/gt/hist.htm, where it is possible to find interesting information on the history of the discipline.

1.1.4 On the main assumptions of the theory

As any theory, also this one needs some basic assumptions to work with. It is outside the scope of this notes to discuss this topic, which really involves several, not only mathematical, aspects. It should never be forgotten that object of the theory is the "rational agent", and that rationality is a central and always in progress issue of philosophy and of the human thought. So I will make only very few remarks, and in the meantime I suggest to take a look at the web page http://plato.stanford.edu/entries/game_theory/.

Just to use only a pair of words, the agent of the theory is supposed to be *egoistic* and *rational*.
What does it mean egoistic in this context? This is probably not difficult to understand. In loose words, we can see that every agent tries to get the best for himself, without taking care of the satisfaction of other agents: for instance, if money is a priority for me, than it is better one dollar to me than 1,000,000 to another person.[2]

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[1] For instance, two people bargaining in principle are both interested in reaching an agreement.

[2] Of course, one could object that it is better to make another have 100,000 dollars and then try to get some beneficial from this. At this purpose, it is important to have always

Just a two lines comment on this. One should avoid to conclude that this theory is unfair, against the ethic and so on. The above assumption does *not* imply that a game theorist will be surprised observing a human being acting in a very generous way. The key point is instead that altruistic behaviour can be a "satisfactory choice" for a person. I can "egoistically" decide that the happiness of another person is my main goal:[3] my generosity will be an effect (maybe a good effect) of this decision taken "egoistically" and efficiently.

Much more complicated is to define how rationality is, and my personal opinion is that at least the fundamentals of the theory do actually deal essentially with trying to clarify this issue. We shall see, in the next chapters, how to analyze different types of games; we shall speak about backward induction for extensive form games with perfect recall, of saddle points for zero sum games, of Nash equilibria in non cooperative models in strategic form, of different solutions in cooperative theory. Well, all these concepts can be considered as a formalization of the idea of rationality, since they suggest an efficient way to play (classes of) games.
However, we need to start with some preliminary, intuitive and simple set of rationality assumptions, and thus I outline now the first basic rationality principles.

The very first point is that each player has some alternatives to choose from, and that the every combination of choices of each player produces a possible outcome of the game. Thus, the very preliminary step is to assume that *each player has preferences on the outcomes.* This means that every player observes all available alternatives (or outcomes) and is able to *order* them, deciding that, for instance, alternative A is better than B, which in turn is better than C and so on. Mathematically, preferences are relations on a nonempty set of alternatives satisfying certain intuitive properties, such as f.i. transitivity. This is a very primitive rationality requirement: *the players are able to order the outcomes of the game according to consistent preferences.*
From a mathematical point of view, working with relations on the set of alternatives is not very convenient. For this reason, usually preferences are expressed by means of a function, called *utility* or *payoff* function, i.e. a real valued function u defined on the set of alternatives, fulfilling the property that $u(A) \geq (>) \, u(B)$ if the alternative A is preferred (strictly preferred) to alternative B. A whole theory has been developed for utility functions. Here we shall take for granted that a utility function is given, i.e. is a primitive object of the model. So, to the player i in general will be assigned a set X_i representing the possible available choices to him, and a utility function $u : \times X_i \to \mathbb{R}$. Observe that the utility function u *does* depend from the choices of *all* players. To understand how crucial can be defining the correct utility functions for the players,

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in mind which model we want to consider. In particular, it is necessary to pay always attention to avoid including underling assumptions, when thinking of the plausibility of the suggestions of the theory. In this case, it is excluded that the other agent could share some benefits with the player from helping him in getting more money. Very often people end to give an incorrect answer to questions of this kind because they do make unconscious assumptions which are not present in the model. Thus when an answer seems to be quite counterintuitive, one should always ask herself whether the model is adequate or not, rather than thinking that the answer of the theory is wrong.

[3] So that a game theorist will never say to her children: Consider how many things I made for you at cost of my time, happiness...

let us consider a famous experiment. It consists of two different lotteries. The first one is the following:

First lottery

Alternative A

gain	probability
2500	33%
2400	66%
0	1%

Alternative B:

gain	probability
2500	0%
2400	100%
0	0%

In a sample of 72 people exposed to this experiment, 82% of them decided to play the Lottery B. This for the theory can be acceptable, even if in general we shall assume that players will follow the expected value rule: in this case, since

$$2500 \times 0.33 + 2400 \times 0.66 + 0 \times 0.01 > 2400,$$

one could infer that the correct choice should be alternative A. However, as mentioned, we can simply assume that for somebody it is preferable to have 2400 with certainty rather than either 2500 with 33% probability and 2400 with probability 66% (i.e. we assume that the utility function satisfies $\frac{34}{100}u(2400) > \frac{33}{100}u(2500)$). Economists speak about *risk aversion* in this case. However, look what happened with the second proposed lottery.

Second lottery

Alternative C

gain	probability
2500	33%
0	67%

Alternative D:

gain	probability
2400	34%
0	66%

At a close look it appears that the two lotteries are the same, since the expected values are the same, apart from a subtraction factor for both of $2400 \times \frac{66}{100}$.

However, in the quoted experiment, 83% of the people interviewed[4] selected lottery C. This shows how delicate is the attribution of a utility function to the players, and a consistency assumption on this point. However discussing this is not a point in Game Theory, where the players do express consistent preferences, usually described by a utility function.

4 Of course the group is the same as in the first lottery.

Always speaking imprecisely, we next assume that the players are *fully able to analyze the problem and take the right decisions.* Again, this sounds at least unclear, but probably an example can shed some light. So, suppose to organize, with a group of people (for instance in a class, during a talk and so on) the following experiment: ask them to write an integer between 1 and 100 (included), *with the aim of guessing the* 2/3 *of the average of all numbers written by the presents.* And you promise a prize to those going closer to this number. There is *only* one rational answer from the point of view of the game theorist, which however, unfortunately, will never win at this game if will write on the paper the correct answer! But let us see how the Game theorist analyzes such a situation. The first step is to observe that it is a nonsense to write a number greater than 2/3 of 100, since the average can be *at most* 100. Since the theory states that the players are rational, it is clear that this line of reasoning will be common to all players. But then, after the first step of the reasoning, this implies that nobody will write a number greater than 2/3 of 100. So that it is a nonsense to write a number greater than $(2/3)^2$ of 100 (second step). But this is what all players know, so that it is a nonsense to write a number greater than $(2/3)^3$ of 100... so that the conclusion for the game theorist is that everybody should write 1. Needless to say, all experiments, even with people acquainted with mathematics, show that you will never win by writing 1. Thus, as a conclusion of this poorly formalized speech, let us say that the player of the theory is a rather idealized person able to consistently express his preferences, and to make a deep analysis of the problem, in the sense that she is able to make the analysis of the behaviour of the other players, and of the consequences of their and her own analysis, and the consequence of the consequence ...

It is time to start to formalize a first concept of rationality in a mathematical fashion. Since the players are willing to get the best for themselves, the following is a *minimal* assumption:

A player does not choose X, if it is available to him Z allowing to him to get more, no matter which choice will make the other player(s).

We speak in this case of *elimination of dominated strategies.*[5]

Before seeing a very simple example of the use of this assumption, let us introduce a convenient way to represent a game.
A (finite) game can be efficiently represented by a *bimatrix* like the following one:

$$\begin{pmatrix} (a_{11}, b_{11}) & \dots & (a_{1m}, b_{1m}) \\ \dots & \dots & \dots \\ (a_{n1}, b_{n1}) & \dots & (a_{nm}, b_{nm}) \end{pmatrix}.$$

Conventionally, the first player chooses a row, the second one a column, and the choice i of the first and j of the second produces an outcome assigning utility (a_{ij} to the first player and b_{ij} to the second.

For instance, the above rule allows for "solving" the following game:

[5] In the sense that x is dominated by z. I am loose on the fact that in the above definition "more" could be mathematically interpreted as either $\geq$ or $>$. As we shall see, this makes a very big difference! Anyway, let us read for the moment the assumption as strictly more or $>$. We shall came back to this point later.

$$\begin{pmatrix} (2,2) & (1,0) \\ (0,1) & (0,0) \end{pmatrix},$$

since the second row (column) is dominated by the first one: thus the only reasonable outcome of this game is the utility pair $(2, 2)$.

In the next section we shall see how this apparently innocuous assumption can produce intriguing situations.

1.2 Game theory versus optimization

One of the first major contributions of Game Theory to a better understanding of the human behavior is to clearly show how some facts that our psychology, used to think in terms of optimization, takes for granted things that could be very inefficient, when interacting with other optimizers. Here are some initial examples.

Is more better than less?

Suppose one person must select how to behave in order to maximize her profit. The profit is quantified by her *utility* function f, so that, from a mathematical point of view, she must find, inside a set X representing her possible choices, an element $\bar{x}$ maximizing f: find $\bar{x} \in X$ such that

$$f(\bar{x}) \geq f(x) \quad \forall x \in X.$$

Now, suppose the decision maker has the possibility to choose between two utility functions f and g, and suppose also that

$$f(x) \geq g(x) \quad \forall x \in X.$$

It is quite natural that she will choose f, guaranteeing to her a better result *no matter what choice she will make.*

Does the same hold when the decision makers are more than one? It is clear that the first intuitive answer is positive! But, let us consider the following example:

Example 1.2.1 The games are described by the following bimatrices. The first one:

$$\begin{pmatrix} (10,10) & (3,15) \\ (15,3) & (5,5) \end{pmatrix},$$

the second one:

$$\begin{pmatrix} (8,8) & (2,7) \\ (7,2) & (0,0) \end{pmatrix}.$$

The two players earn more, *in any situation*, i.e. in every possible pair of joint choice, in the first game. Easy to guess they will decide to play the first one, without any doubt. But, *surprise*, the rationality axiom implies that the final outcome is better, *for both*, in the second game.

Is it better to have less or more options ?

In (one player) decision theory the answer is absolutely obvious: on a bigger domain, the maximum value of the utility function cannot be less. Does the same happen when the players are more than one? Look at the following example.

Example 1.2.2 Consider the following two games. The first:

$$\begin{pmatrix} (10,10) & (3,5) \\ (5,3) & (1,1) \end{pmatrix}.$$

The second, containing all possible outcomes the first, and some further outcomes, is the following:

$$\begin{pmatrix} (1,1) & (11,0) & (4,0) \\ (0,11) & (10,10) & (3,5) \\ (0,4) & (5,3) & (1,1) \end{pmatrix}.$$

By using the rationality axiom we see that the outcome in the first is 10 for both, in the second 1 for both: once again, a non intuitive result. It is true that in real life situation we see people that have difficulties in choosing among (too) many alternatives, even when they act alone, but our intuition is that this is due to psychological reasons. In this example instead we see that, when interacting, even perfectly rational agents can have a damage if other possible choices are given to them.

A comment on the rationality assumption made before. It is a very weak assumption, and we can agree that for interesting games it will not be sufficient to forecast its evolution. In interesting cases, my choice between two options will depend also from the choice of the other player: otherwise, the game is not interesting![6] Nevertheless, with this apparently very weak assumption, we are already able to produce surprising examples. In the next section, we shall consider other similar situations, where interaction between players produces complicated issues.

1.3 Bad surprises

We start by considering two issues. The first one deals with existence of multiple optimal choices, the second one deals with something already mentioned before, i.e. the fact that one must be very careful about eliminating strategies which are dominated in a weak sense.
About the first point, let us observe that a decision maker acting alone does not care about multiplicity of optimal solutions. Mathematically, apart maybe from the point of view of the explicit calculus of a solution, having more than one is not painful. What really matters the decision maker is the *maximum value* of its utility function: if it is taken at several points just means that he has several (equivalent) optimal policies. It is an easy matter to understand that the same does not apply in interactive setting. To see this, we consider a celebrated example, called in literature the *battle of sexes*:

A bimatrix describing such a situation could be the following:

[6] An uninteresting game usually is not a good model for real situations...

Example 1.3.1

$$\begin{pmatrix} (10,0) & (-5,-10) \\ (-10,-5) & (0,10) \end{pmatrix}.$$

It describes the situation of two lovers deciding where to spend the evening, say either at the stadium or at the theater. Both love to be together, but they have different first options.
Observe that our preliminary rationality rule cannot provide an outcome of the game, since no row (column) is dominated. However, taking for granted that such a game should have an equilibrium, it is clear that this cannot be one of the outcomes $(-10,-5)$, $(-5,-10)$.[7] Furthermore, given the symmetry of the situation, it is also clear that it is not possible to distinguish between the two outcomes $(10,0)$, $(0,10)$. However, it is quite clear that the players are not indifferent with respect to these outcomes: the first one likes better the first outcome, obviously the contrary is true for the second player. This creates several problems: one of them, for instance, is that in any case implementing one of the two equilibria in the game requires *coordination* between the players (and it is not difficult to imagine a situation when communicating for both is impossible).

About the second issue, i.e. dealing with weakly versus strongly dominated strategies, to begin with let us say that mathematically, a choice $\bar{x} \in X$ for the first player, having f as a utility function and X as a space of choices, is *weakly dominated* if there exists $x \in X$ such that

$$f(\bar{x}, y) \leq f(x, y)$$

for all $y \in Y$, the set of choices of the second player.[8] On the contrary, a choice $\bar{x} \in X$ is *strictly dominated* if there exists $x \in X$ such that

$$f(\bar{x}, y) < f(x, y)$$

for all $y \in Y$, the set of choices of the second player. There is no doubt that a strictly dominated strategy must be eliminated, but what about a weakly dominated one? Let us consider the following example.

Example 1.3.2 Suppose three people have to vote among three different alternatives, say A, B, C, in order to select one of them. They vote, and if there is an alternative getting at least two votes this will be the decision. Otherwise the alternative voted by the first player will be winning. The preferences of the three players are as follows:

$$\begin{array}{ccccc} A & \succ & B & \succ & C, \\ B & \succ & C & \succ & A, \\ C & \succ & A & \succ & B. \end{array}$$

If we use the procedure of eliminating weakly dominated strategies, it can be shown that the first player will vote A, since it is weakly dominant, while the two other players will eliminate A and B respectively, which are the worst choice for them,

[7] They want to stay together, so we are excluding the options of being separated.

[8] This can of course easily extended to games with more than two players.

and thus weakly dominated. At this point the first player is not anymore in the game, while the other two face the following situation (verify it):

$$\begin{pmatrix} A & A \\ C & A \end{pmatrix},$$

where the second player must choose one row, the first one representing his choice of B, while the first column represents the choice of C for the third player. At this point the solution is clear: since both the second and the third players like better C than A, the final result will be C.

What is interesting in the above result is that the game has an apparently stronger player, the first one, and the final result, obtained by a certain (reasonable) procedure is what he actually dislikes the most. Observe that the result A too, his preferred, can be supported by an idea of rationality (we shall see it later). But this is not my main point: what is interesting to observe is that the procedure of deleting (weakly) dominated strategies can be dangerous.

But in some sense, the troubles we have seen so far are (almost) nothing, compared with what is shown by the following example:

$$\begin{pmatrix} (1000, 1000) & (0, 1500) \\ (1500, 0) & (1, 1) \end{pmatrix}.$$

Solving this game by means of the rationality rule we see that the unique rational outcome of the game provides 1 to both. But this result is a nightmare! For, our rational players have the possibility to both get 1000 but they will have only 1![9] Individual rationality seems to go in the opposite direction with respect to collective welfare. But this is a long and well known story. It is quite evident that living in a society can be profitable for everybody. A society however needs rules, and the respect of the rules guarantees (or should guarantee) social welfare. However, from a purely individual point of view, breaking the rules is convenient. This is the very leading idea developed by the philosopher Hobbes in his Leviathan, and the proposed solution is to create the figure of the dictator, which is a person with the authority to make binding the agreements by punishing those not following the rules. Fortunately Game Theory can in a sense provide (partial) results showing that a more sophisticated approach to the problem can support the idea that collaborating can be a rational behavior, but it must be clear that there is no hope to cope with this aspect just by changing the initial rationality assumption: it is better to develop also different models.[10]

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[9] It should be observed that the use of the numbers 1000 and 1 to underline the big loss related in this game with elimination of dominated strategies once again appeals to psychological aspects not entering in our approach. Nothing changes, form the point of view of our assumptions, to have 1.01 instead of 1000 in the bimatrix. What really matters is the ordering relations between numbers. In other words, nothing changes, f.i. if we apply monotone transformations to the utility functions.

[10] In this case introducing repeated games dramatically changes the situation. And in some sense games are "almost always" repeated.

To conclude this section, let us see, without comments, some situation leading to a bimatrix similar to that one above. Though the examples are very naive, they show that the prisoner dilemma situation is not only an abstract case study.

Example 1.3.3
1. Two people are suspected of being responsible of a serious crime. The judge makes them the following proposal: for the crime I suspect you are responsible, I can sentence you 7 years in jail. If both confess however I will appreciate this and give you only 5 years. If one confesses (the responsibility of both) and the other does not, the collaborator will be free. If no one confesses, I will sentence you for 1 year of jail, since I do not have enough evidence for a Jury that you are guilty (even if I am sure about this), but I can prove that you are guilty of a lesser crime. This proposal is submitted to both
2. Do you like better if I give 1 euros to you or 100 euros to your roommate?
3. Two nuclear powers must decide their policy. Both like to be superior to the opponent, both know that nuclear arms are very expensive
4. Two coffee bar must decide, at the same time, the price of the cappuccino (between two possible choices). Having a lower price with respect to the opponent is very fruitful, since the gain in the number of customers covers the lesser gain for the lower price. Having the same price means sharing the same number of customers
5. Two animals compete for the same territory: they can choose to be either aggressive or submissive. Having superiority on the other is a priority, but to be aggressive costs a lot;
6. Two students are preparing a test. Who makes the test better, gets the maximum result. They can choose whether to work a little or a lot. Being the best at the test is the most desirable outcome for both.

1.4 Exercises

Exercise 1.4.1 The Prisoner Dilemma (Example 1.3.3) Two people are suspected of being responsible of a serious crime. The judge makes them the following proposal: for the crime I suspect you are responsible and I can sentence you 7 years in jail. If both confess, however, I will appreciate this and give you only 5 years. If one confesses (the responsibility of both) and the other does not, the collaborator will be free. If no one confesses, I will sentence you 1 year of jail, since I do not have enough evidence for a Jury that you are guilty (even if I am sure about this), but I can prove that you are guilty of a lesser crime. This proposal is submitted to both.
Draw the matrix of the game and discuss the possible results.

Solution The game is described by the following bimatrix

$$\begin{pmatrix} (5,5) & (0,7) \\ (7,0) & (1,1) \end{pmatrix}$$

where the first row and the first column represent the choice "confess" and the second ones the choice "not to confess". Player I will not play the second row because of the rationality rule. In fact, playing the second row if player II decides to confess I should spend 7 years in gale instead of 5 and if she decides not to confess 1 year instead of 0; we say that the second row is dominated by the first one. Similarly player II will

not play the second column (we can notice that the game is symmetric).
The outcome is they both confess and spend 5 years in prison (but we notice they had the opportunity to spend only 1 year each in prison!).

Exercise 1.4.2 The chicken game Two guys are driving in opposite directions on the same narrow road. They can decide if they want to deviate (on their own right) or not. Each one of them would like to go straight on, but for sure they do not want to crash!
Draw the matrix of the game and discuss the possible results.

Solution Here we obtain a different situation, given by the following bimatrix

$$\begin{pmatrix} (-1,-1) & (1,10) \\ (10,1) & (-10,-10) \end{pmatrix}$$

the first row and column represent the choice "deviate", while the second ones the choice "go straight on". The outcome $(-10,-10)$ represents their utilities when they crash, $(-1,-1)$ when they both deviate, i.e. they do not crash, but they are a bit disappointed because they had to turn uselessly. The other two outcomes represent the situation in which they do not crash and the one who went straight on is very satisfied.
Differently from the previous exercise, we cannot implement the rationality rule, as no row/column is dominated. We can just simply notice $(-1,-1)$ and $(-10,-10)$ are not the solutions, but we cannot decide between the other outcomes. This situation is similar to the Battle of Sexes game.

Exercise 1.4.3 A game of matches 4 matches are on a table. The first player has to decide to take 1 or 2 matches, then the second one has the same choice and so on. The last one to take a match loses. Is it better to be the first or the second player?
What if the 4 matches are divided in 2 groups and each player can decide to take 1 match or 2 matches from the same group?

Solution It is better to be the second player as, playing a good strategy, she always wins. If player I takes 1 match, II takes 2 of them, letting the last match to player I and making her lose. If player I takes 2 matches, II takes only one of them, letting again the last one to player I. The tree of the game is shown in Figure 1.1.

We can notice that in the second situation player II still wins, using the same strategy than in the first case. The tree of the game, anyway, is a bit different, as it is shown in Figure 1.2.

Exercise 1.4.4 The Russian Roulette Two players have a revolver each with one shot out of six in it. Every player puts 1 euro at the beginning. The first player has to decide whether to play or to pass the revolver to the second player, he adds 1 euro in the first case and 2 euros in the second one. If the first player is not dead, the second one plays with the same quotas and the same probabilities and then the game is finished. At the end if they are both alive, they share the money, otherwise the survivor takes all. Represent the tree of the game and the outcomes of the players.

Solution The tree of the game is shown in Figure 1.3. As it is a zero sum game (i.e. a game where what one player gets is the opposite of what the other player gets, in any outcomes of the game), only the final outcomes of player I are written.

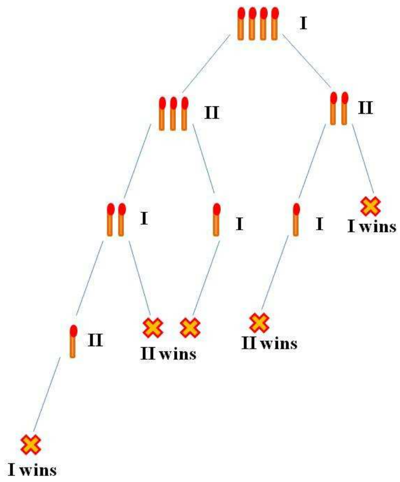

Fig. 1.1. Exercise 1.4.3

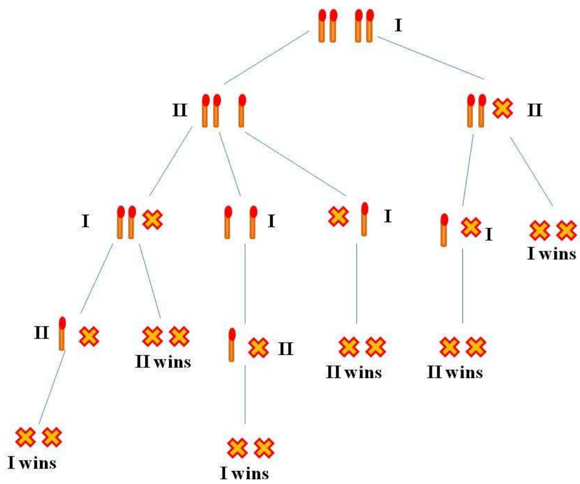

Fig. 1.2. Exercise 1.4.3

Exercise 1.4.5 Beauty contest A group of people is asked to write on a paper, together with the name and the surname, a natural number n, $1 \leq n \leq 100$. The one who will turn out to have written the closest number to the $\frac{2}{3}$ of the average of all the numbers is the winner. It is possible there is more than one winner. What would you write on your paper?

Solution We notice that the average M of k natural numbers n_k with $1 \leq n_k \leq 100$ is s.t $M \leq 100$. But

$$\frac{2}{3}100 \approx 66.7$$

it is indeed a nonsense to write a number bigger than 67.
Now we notice that the average M_1 of k natural numbers n_k with $1 \leq n_k \leq 67$ is s.t.

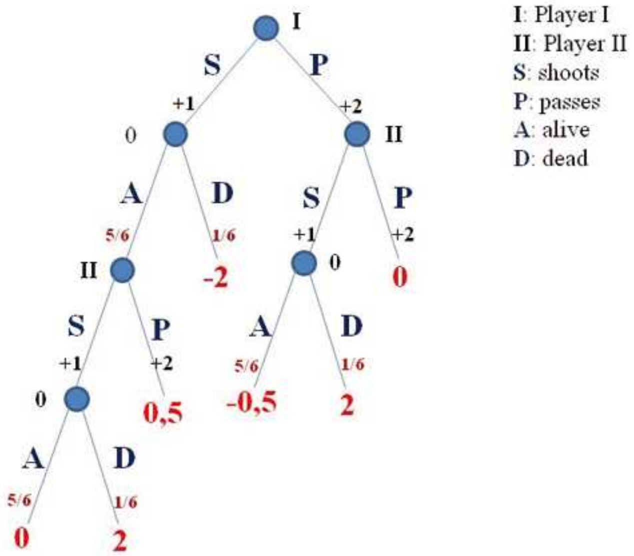

Fig. 1.3. Exercise 1.4.4

$M_1 \leq 67$. But

$$\frac{2}{3}67 \approx 44.6$$

By the process we arrive to notice that the optimal strategy is to write 1 on the paper. It is however very rare that in a real situation all the players arrive to iterate the process more than once or twice and the player who writes 1, normally, is not going to win.

Exercise 1.4.6 Playing cards Two players, A and B, start putting 1 dollar each. Player I takes a card, without showing it to II, and she decides if she wants to stop or to rise 1 dollar. Player II decides if she wants to see or not. To see she has to add 1 dollar also. If I stops, I wins only if the card is red. If I rises again, I wins if the card is red or if II decides not to see. Draw the tree of the game.

Solution We represent the tree of the game in Figure 1.4. We notice that when player II has to play, she does not know if player I has taken a red or a black card, then she does not know if she is playing in the left part of the tree or in the right part. This means that the information is incomplete and we represent it by a dotted line.

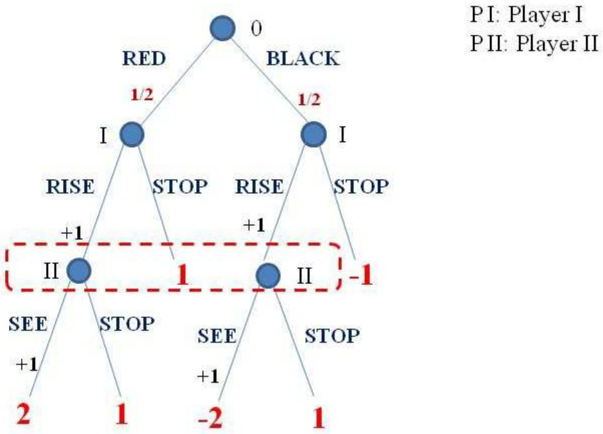

Fig. 1.4. Exercise 1.4.6

2

Games in extensive form

2.1 Games in extensive form

By *game in extensive form* we intend the mathematical formalization of all relevant information about the game. This means in particular to specify:

- the initial setting
- all possible evolutions (i.e. all allowed moves at each possible situation)
- all final outcomes of a game, and the preferences of the players on them.

Usually, this is made by constructing *the tree* of the game. We see this in the next example.

Example 2.1.1 There are three politicians (the players) that must vote whether to increase or not their salaries (*Initial situation.*). The first one publicly declares his vote (Yes or No), then it is the turn of the second one, finally the third one declares his move. The salary will be increased if at least two vote Y (*All possible evolutions.*). This information allows constructing all possible evolutions of the game. Finally, we need to specify preferences of the players. We assume that they all have the same preferences: in increasing order, voting Y and do not get more salary (a nightmare: no money and a lot of criticism by the electors), voting N and do not get more money, voting Y and get the money, voting N and get the money (very nice: looking unselfish and getting the money!) Let us say that their level of satisfaction is 4, 3, 2, 1 respectively (*Preferences of the players on the outcomes*). Observe that the game is intriguing since the players want to get something but at the same time would prefer to declare the contrary. Here is the tree of the game in Figure 2.1.

This is simple to understand, and an efficient description of the game. Observe that in the above game every player knows exactly at any time what is the current situation, the past history, all possible future evolutions. Such games are called *perfect information games.*

The next game presents a situation that could be considered a little different.

Example 2.1.2 The players must decide in sequence whether to play or not. If both decide to play, a coin is tossed, and the first one wins if the coin shows head, otherwise is the second to win. In Figure 2.2 is the game tree:

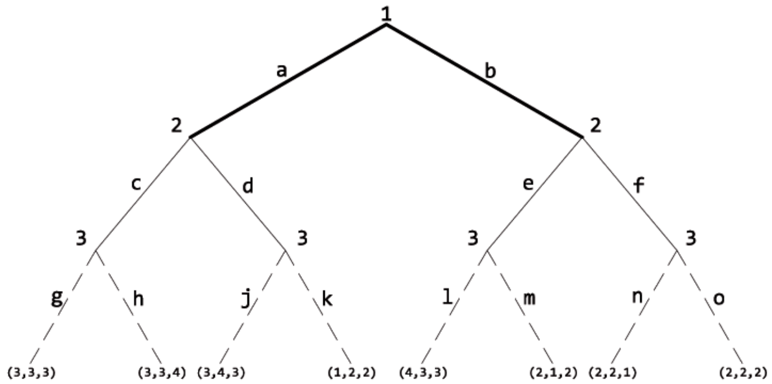

Fig. 2.1. The game of the three politicians

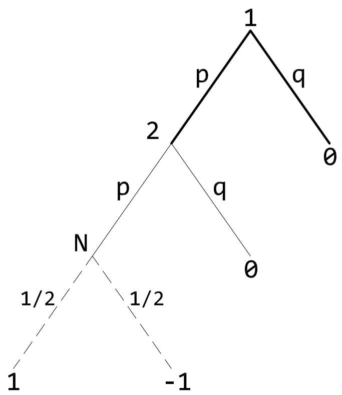

Fig. 2.2. Example 2.1.2

Where is the difference between this example and Example 2.1.1? Here the chance (N is used to label it) plays a role. This maybe changes things from a psychological point of view, but not from the point of view of a game theorist. For, rationality assumes that the players will evaluate their utility functions in terms of *expected utilities*: winning 2 with probability 1/3 and loosing 1 with probability 2/3 is absolutely the same as having 0 with probability one (we are assuming here the satisfaction for winning one is the same as the dissatisfaction of loosing one, a kind of risk neutrality). This allows players having no uncertainty about past moves (of other players) and possible future evolution, so that this kind of games are of the same nature as the previous one. Thus practically the above game is equivalent to the following one in Figure 2.3:

which looks totally uninteresting, but still has perfect information. Thus the presence of the chance in the game does not imply necessarily that the information is not perfect: the key point being that the chance moves must be observed by both players.

A different situation is, for example, if the game rules require contemporary moves to the players. It is the case of the battle of sexes or of the familiar stone, scissor,

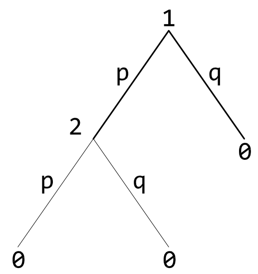

Fig. 2.3. Simplified Example 2.1.2

paper game. This game is still called of *complete information*, since the features of the players (strategies, utility functions) are common knowledge. Nevertheless the situation is different from before, since the second player does not know the decision of the first, once is called to decide.[1] In this case it is still possible to use the tree structure to describe the game, even if it is in this case less meaningful. We need however a visual device to signal that the second player decides without knowing what the first one did at the first stage. Here (Fig 2.4) we show the prisoner dilemma tree, where the branch on the left indicates the strategy of confessing, that one on the right of not confessing. Payments are in years of jail. Observe that in this case calling the players player one and player two is a convention, not indicating who is moving first.

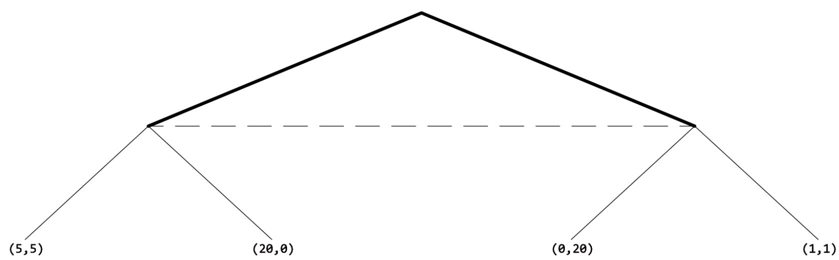

Fig. 2.4. The prisoner dilemma.

More interesting, perhaps, is to describe the situation of a prisoner dilemma game repeated twice, with the two players knowing the result of the first game. The following picture (Fig 4.1), showing only the tree without further explanation, should be clear.

Observe that we put a dotted line connecting the nodes relative to the second player; this has the meaning that the second player knows to be in one of the two nodes, but *not* in which one of the two nodes. The set of the two nodes is called an *information set* (for the second player).

[1] Clearly, in case of two players and contemporary moves we can assume that one player is the first to decide, and that the other one decides without knowing the decision of the first one. And this can clearly be extended to the case of more players.

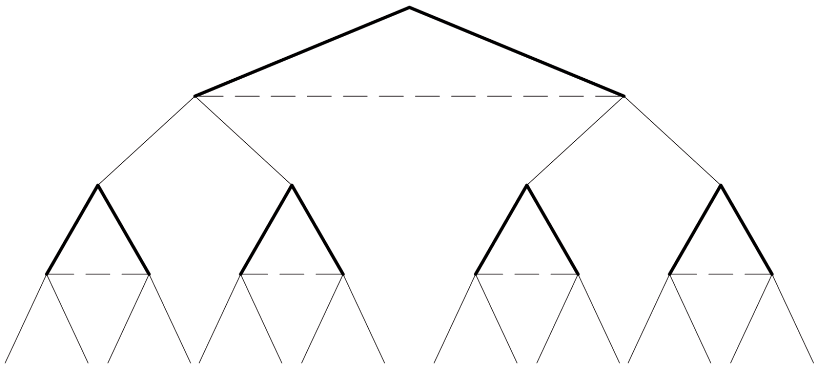

Fig. 2.5. The prisoner dilemma repeated twice.

The complete information case does not describe several situations. Actually, it is more frequent the case when the players are *uncertain* about some other players' features. Let us see a simple example of this.

Example 2.1.3 A firm must decide whether to enter or not in a market where there is only one other firm. This last one, called the monopolist, can decide whether to fight or not the entrant. The monopolist could reasonably be of different types, for instance it could be a *weak* or *strong* monopolist. In the case it is weak, it does not like to fight, otherwise it will fight the entrant.
We can summarize the situation in the following tables, where the line player, i.e. the first player, is the entrant. The first bimatrix represents the weak monopolist, the first row represents the decision of the entrant to actually enter into the market, while the first column represents the choice of the monopolist of not fighting the entrant:

$$\begin{pmatrix} (1,0) & (-1,-1) \\ (0,2) & (0,2) \end{pmatrix}, \begin{pmatrix} (1,-1) & (-1,0) \\ (0,2) & (0,2) \end{pmatrix}.$$

Observe that we put in the second line the decision of the entrant of not entering in the market, and on the first column the decision of the monopolist of not fighting. In this case, to analyze the game we need to assume that the entrant has some ideas on the type of the monopolist. We assume that he attaches probability p to the weak monopolist. The game tree can be built up by imagining that the Nature makes a choice on the type of the monopolist, choice which is obviously not seen by the entrant but known to the monopolist.

And here in Figure 2.6 is the tree of the game.

These games, as intuitive, are more complicated to be analyzed. In a subsequent chapter we shall see how it is possible to tackle the problem. Observe also that in this type of games the idea of repetition is very important, since observing the behavior of the players can give some more information: a monopolist, in a repeated game, could decide to fight, even if it is of the weak type, in order to *signal* entrants that he will fight also in the future. This could discourage entrants and give him a global advantage, even if at the stage he fights he does not take an optimal decision.

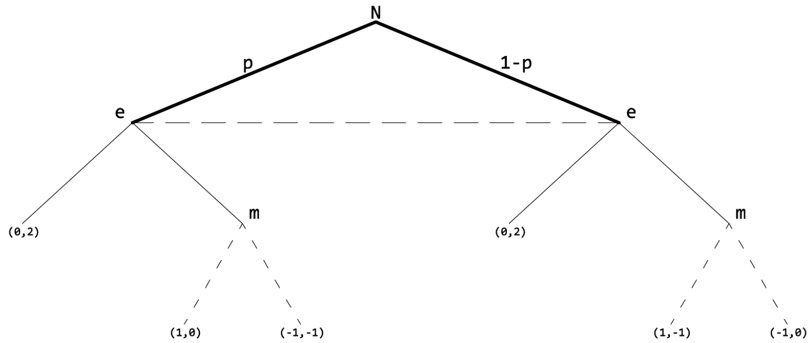

Fig. 2.6. The monopolist game.

Let us now summarize, in an informal way, what we need to describe a game by means of a game tree.

The game in extensive form is characterized by nodes or vertices, and branches (i.e. pairs of nodes); we shall say that a vertex w *follows* the vertex v if there are $v_1, \dots v_k$ nodes such that $v_1 = v$, $v_k = w$ and there are branches connecting v_i to v_{i+1} for all i:[2] we shall denote by R_v the set of branches having v as upper node, and by R the set of all branches.

1. There is an element $v_0 \in V$ such that, if $u \in V$, then u follows v_0; v_0 *represents the initial situation in the game: e.g. the positions of chess in the board*
2. A partition $V_1, \dots, V_n, V_N, V_T$ of V is given; V_i *represents the set of vertices where the player i is called to play,* V_N *the vertices where there is a move by the Nature,* V_T *are the terminal nodes, representing the final situations in the game*
3. For every $v \in V_N$ there is a probability distribution on R_v
4. To each $v \in V_T$ is associated a n-tuple of real numbers; *the utilities associated to the players in the final situations*
5. Each set V_i, $i = 1, \dots, n$ is partitioned in sets W_i^j, $j = 1, \dots, j(i)$, such that:
   - $v, w \in W_i^j$ implies that there is an isomorphism between R_v and R_w by identifying branches according to an isomorphism we shall denote by R_i^j the set of branches coming out from W_i^j.[3]
   - $v, w \in W_i^j$ implies v does not follow w and w does not follow v; *Player i knows to be in* W_i^j *but does not know in which precise node actually is. The two above conditions establish consistency.*

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[2] The branches are oriented, thus v_1 is the upper node of the branch connecting v_1 to v_2 ... Saying that w follows v means that the game allows moves bringing it from the situation described by w to the situation described by v.

[3] This can sound rather obscure, but in reality is simple. Think f.i. to the tree of the prisoner dilemma. We need to put a branch representing the fact that the second player does not confess, coming out from both possible nodes corresponding to the decision confess/not confess of the first player. Thus we have two different branches representing the same decision. The isomorphism connects branches following different nodes in the same information set and representing the same move of the player.

A *play* (or *path*) is a chain $(x_1, y_1), \dots, (x_k, y_k)$ such that:

- there is a branch connecting x_i to y_i for all i;
- $x_1 = v_0$;
- $y_i = x_{i+1}$ for $i = 1, \dots, k-1$;
- $y_k \in V_T$.

The *length* of the game is the cardinality of the longest path of the tree.

Observe that a game is of perfect information if and only if the sets W_i^j contain a unique element (i.e. they are singletons) for all i, j.

Now that we have found a useful representation of a game we proceed in seeing how this can help in the analysis of the game itself. To do this, let us come back for a moment to the first game of the three politicians. The game tree clearly shows a very intuitive fact: notwithstanding the players must make the same moves (saying Yes or No), they are in very different situations. In fact, the first player has to take a decision at a node, the initial one, while for instance the second one must decide his behavior *depending* from what the first one did, i.e. at two possible nodes. Thus, in his (a priori) analysis of the game, he must decide what to do in case the first one choose Yes and what to do in case he said No. And the third player needs to specify what to do in four different situations. Remember however that saying that they are in different situations does not mean at all that their analysis will be different. It is a basic assumption of the theory that each players analyzes everything, including of course the possible behaviour of the other players. And so their analysis and their conclusions will be identical. So when we shall say the first player thinks ... we mean that we are analyzing his behavior, but the relative analysis will be made by all players.

It is clear that a specification of an available move for all players at all information sets labelled with their names necessarily determines a specific outcome of the game. This explains the importance of the next definition.

Definition 2.1.1 *A* pure strategy *for the player i is a function*

$$f : \{W_i^j : j = 1, \dots, j(i)\} \to R,$$

such that $f(W_i^j) \in R_i^j$. *A* mixed strategy *is a probability distribution over the set of the pure strategies. A* strategy profile *is a list of strategies, one for each player.*

Thus a pure strategy specifies the choice of the player at each information set labelled with the name of the player, even those not reached by the actual play of the game because of a former choice specified by the strategy itself. We shall come back to this important point later. Instead, the definition of mixed strategy assumes a more intuitive meaning if we think of several repetitions of the same game. It is not hard to imagine that in several games, played repeatedly, it is not a good idea to play always the same strategy (think f.i. playing the stone, scissors, paper several times with the same person, you will never use the same strategy at every round of the game!). We shall discuss this later.

As mentioned before, the game can be given another form, which can be useful as well, for instance when we are not interested in analyzing the dynamic of the game. From the description of the game in extensive form, we are able to derive all strategies for the players. Suppose for the moment there are only two players, this makes notation easier. Thus, we are given two sets of strategies for the players, say X and Y, called the strategy spaces. Then, as already remarked a specification of a pair (x, y), where $x \in X$, $y \in Y$, provides a unique outcome of the game and thus a payoff associated to the players.[4] Thus we are also given two functions $f, g : X \times Y \to \mathbb{R}$, representing the payoffs of the players. Thus we can define a two player game in *normal* or *strategic* form is a quadruplet $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$.

In the case of a finite number of strategies for the two players, an efficient way to describe the game is, as we have seen, to put payoffs in a bimatrix. In case $g = -f$, we shall write only the payments of the first player, and the bimatrix becomes a matrix.

The two forms for describing games have advantages and disadvantages, in any case are both very useful. It is clear that from the extensive form it is possible to produce the strategic one.[5] However, it is always a long and sometimes difficult task to pass to the strategic form, unless the game is very simple. We shall discuss this issue later.

2.2 Games of perfect information

It is time to put again our attention to the class of games of perfect information, since it is not hard to see, by means of their description as a game tree, how to efficiently analyze them; moreover historically they were the first to be considered. Let us come back to the Example 2.1.1. The method to understand how rational players will play it is called *backward induction.* It means that the game must be analyzed by starting *from the end*, rather than from the beginning.

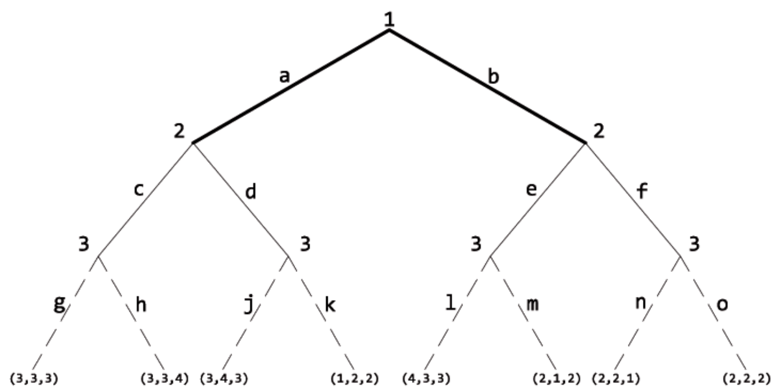

Fig. 2.7. The game of the three politicians

Let us try to understand what will happen at every terminal node, i.e. a node such that all branches going out from it lead to a final situation. In our particular game all

[4] To games where the alternative are win, draw, loose, we can for instance attach to them, conventionally, the values 1,0,-1, respectively.

[5] The other way around is less interesting.

these nodes are attached to the third voter. They are labelled in figure by the digit 3. At the first node labelled with 3, the player observes that if he chooses Y (branch labelled by g) he gets 3, otherwise he gets 4. Thus he will choose to say N. I leave as a simple job to check what will happen at all other nodes. What is really important is that the third player knows what to do at every node he is called to make a decision, *and the other players know what he will do.* At this point, the second voter is able to decide for his best, at nodes labelled with 2. For instance, you can check that at the first one he will say N. It is clear that, by doing this, we are able to arrive at the top, and to know the exit of every game of this type. In our example, you can check that the first voter will vote against the increase of the salaries, while the other ones will vote to have it. Think a little about it. It is an easy intuition to think that the final result of the process will be that the three guys will be richer: they all love it better than having the same amount of money. And, with a little more thought, we can understand that it is logical that the first one will vote against the proposal, because in this case he will *force* the two other ones to vote for the increase of the salaries. But if you ask a group of people to say what they think it will happen in such a situation, several of them will probably answer they would like to be the last voter, thinking, erroneously, that the first two will vote for the money because they want the money.
This simple example induces to think that games described by such a tree structure are always solvable. True: we are systematically applying an obvious optimality rule (everybody, when called to decide, makes the best for him), and we are able to arrive to a conclusion. Observe that very popular games fit in the above description, think of chess and checkers, for instance. This fact imposes a remark. Since backward induction implies that the game has always the same outcome for the players, why do people play such games, and why the result is not always the same? Once again, it must be taken into account our rationality assumption, which implies in particular that players are fully capable to analyze the game. In this context, this means that they are able to construct the tree of the game, and to apply the backward induction. Now, it is intuitive that *once* we have the tree of a game we are able to apply to backward induction, what is less obvious is to be able to construct the tree of the game! To be clear, nobody (not even the most powerful computer) will ever construct the tree of the chess. The key point is that even for very simple games, with few moves and easy rules, it is out of question to think to be able to explicitly write down the tree of the game. Actually, what is so interesting in these games is exactly the fact that the good player has at least an intuition that certain branches must not be explored, since it is very likely that they will cause trouble to the player. The domain of artificial intelligence is of course deeply involved in such a type of questions, the fact that IBM spent a lot of money to create computer and programs able to beat the human being in a series of chess games is perfectly rational...

An interesting question is whether the backward induction always provides a *unique* outcome of the game. The following example shows that it is not the case.

Example 2.2.1 The game goes as follows. I have two identical precious objects and I tell to player one how many of them wants to keep for himself; the other one(s) will go to the second player, but only at the condition that he agrees with the decision of the first. This last one has of course the possibility to no object, one or two. There are two possible outcomes, according to backward induction: the first player will have either one object or both. The point is that, when he give me both, the second player is indifferent between the two options of saying yes/no.[6]

So, once again we see that being indifferent among several alternatives can make difficult finding a "solution" of a game.

One more question. Suppose there is a unique outcome (it is easy to imagine sufficient conditions for this). Can we be sure in this case that the outcome provided by the backward induction is completely satisfactory? I mean, a situation like the prisoner dilemma, where individual rationality forces a poor social result, can occur also in games to which the backward induction applies? Look at the following, very famous, example.

Example 2.2.2 (The centipedes) Two players play the following game.

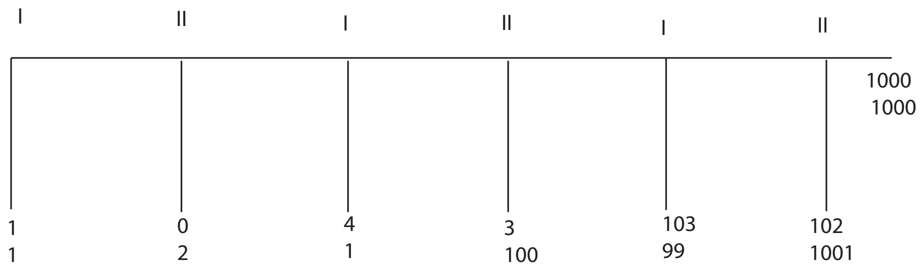

Fig. 2.8. The centipedes

It is quite clear that the outcome is very disappointing for both.

2.3 The Zermelo theorem

2.3.1 About the definition of strategy

Let us comment now on the definition of strategy. As we have seen, a strategy for a player is the specification of a choice in every decision node labelled with his name. This implies that a strategy profile necessarily singles out a precise outcome of the game. So that the definition is very useful, but *only* for theoretical reasons. First of all,

[6] A variant of this game, also called the ultimatum game, has been used in numerous experiments. Typically, to player A is given an amount of money, f.i. 100 Euros, and asked to make an offer to a player B, unknown to him. And player B can either refuse (in this case they receive nothing) or accept. On average, an offer of less than 30 Euros is refused, and it is shown that some parts of the brain are more activated in this case. The very delicate point is to accurately specify the utility functions of the players. It is not obvious to infer that a player refusing an offer of 25 Euros is irrational: probably, her satisfaction to punish player A is superior (at least at a first reaction) to the satisfaction of having 25 Euros. The study of these types of questions is very interesting, but goes outside the field of mathematical game theory.

it is obvious that, even for very simple games, listing all strategies can be a difficult and boring task. Secondly, it is quite clear that, once a first move is made by a player, some strategies appear useless to be considered. Let us pause for a second, and look at the following picture (Fig 2.9).

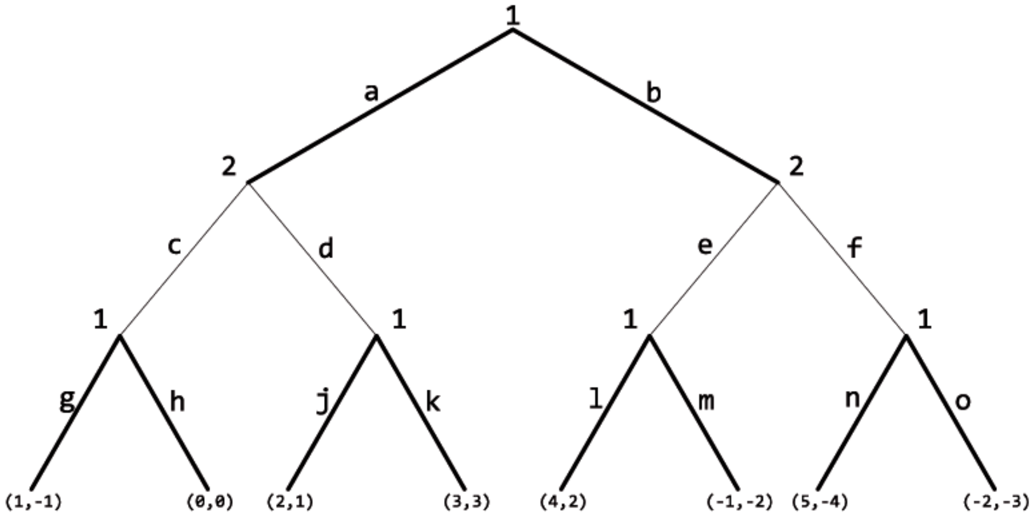

Fig. 2.9. Counting strategies

There are five nodes labelled player one (one node is the root). At the root, she has a choice between two branches, labelled a, b. The second player is now called to make a move, then it is again the turn of the first player, and at this point the game ends. There are eight final nodes, and attached to each them is a pair of numbers, the first (second) one giving the payoff of the first (second) player.

How many strategies does the first player have? An easy calculation shows that they are 32 (the number of functions defined on a set of five elements, and such that to each element is possible to associate two choices). It should be clear what we denote by (a, g, j, m, o). It should be clear as well that this strategy will not produce a different outcome from the strategy (a, g, j, l, n), no matter the second player does! What we want to point out is that once the first player has chosen the branch a at the root, it is irrelevant the specification of her choice at the two nodes reached only in the case her first choice would be b!

However, we cannot conclude from this that the definition of strategy does not work, as it is. First of all, it is a basic concept in order to develop the whole theory. Secondly, the analysis at every point of the tree provides in any case information to the players. In particular, to apply backward induction the choice of every player at every node must be analyzed. This selects specific outcomes with properties that are not shared by other solution concepts (given in a more general context), making the backward induction procedure, whenever available, a strong solution concept. For, it can be considered as an *hyper rational* request of behavior, since it requires that the players act optimally even in parts of the trees which are not reached, once the selected strategies are implemented. In our example, the final outcome is $(4, 2)$, the optimal strategy for player one is (a, g, k, l, o), prescribing to her an optimal behavior also, for instance at the node joined by the move c of the second player, which is not reached when the first one uses her optimal strategy. We insist on this point since we shall see that in other situations (games in strategic form) it can happen that in some equilibrium situation the players are supposed to act not optimally (at branches that will *not* be reached according to the the moves of the players, otherwise this would go against any assumption of rationality).

2.3.2 The chess theorem

Let us point out once again what we can state about backward induction. Calling *path* (or *play*) any series of possible consecutive moves (for instance *ach* is a path in the game above), we can define *length* of a game the number of branches of its longest path. Remember that we are dealing with finite trees (games). It is clear that we are able to find a solution for games of length one. These are degenerated games, since the decision maker is only one, and by rationality he will select the optimal result for him (he must find the maximum of a finite collection of numbers). From this, we are able to solve games with length two. For, what backward induction says is that player one, at the root, will select the branch giving him the highest payoff. She is actually able to know the payoffs attached to the branches, since either they are terminal branches, thus a payoff is automatically attached to it, or it goes to a node, but in this case the node is the root of a game of length one, whose solution is known from the previous step.

Clearly, this is a procedure based on induction, and clearly we are now able to understand how to pass from a game of length n to a game of length $n + 1$.

Observe that the above procedure can be carried on because of two fundamental features of these game trees:

- they are of finite length;
- taking any node x, what follows it is still a game tree for which x is the root.

These are the conditions making possible the procedure of the backward induction.

Let us see, to make some practice, what kind of result we can state for these games, taking as an example the game of chess. Here is the result:

Theorem 2.3.1 *In the game of chess, it holds one and only one of the following alternatives:*

1. *The white has a strategy making him win, no matter what strategy will use the black*
2. *The black has a strategy making her win, no matter what strategy will use the white*
3. *The white has a strategy making him to get at least the draw, no matter what strategy will use the black, and the same holds for the black.*

As a comment, we show in the following pictures (Fig 2.10) the three different situations, imagining (power of our minds!) to have solved, by means of the backward induction, all subgames starting from the nodes joined by branches leaving the root. At the place of the subgame, we put in the picture the result of the game.[7]

[7] b (w) means that the black (white) wins, d means that the result is a draw, and dotted lines mean that in between the two branches there are many other ones.

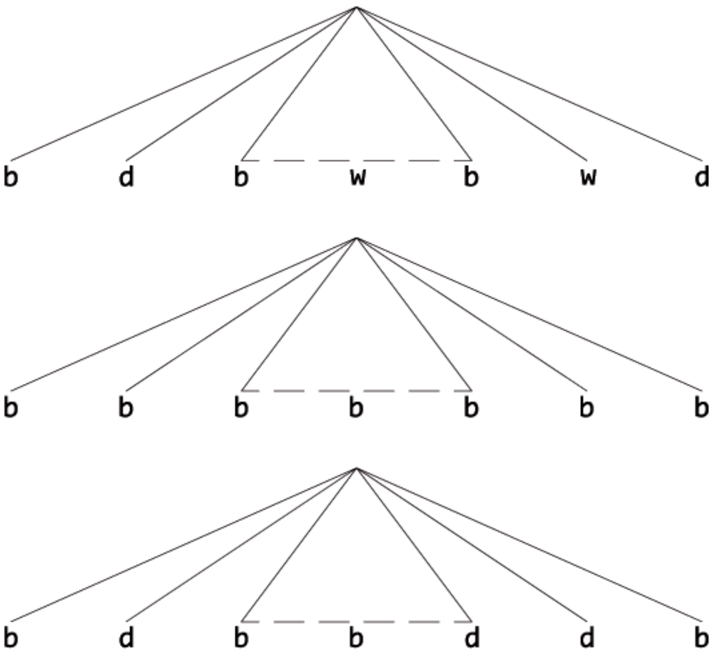

Fig. 2.10. The possible chess situations, after the first move of the white. Only winners in the subtrees are indicated; the three possible situations are displayed.

To conclude this section, observe that in games where the tie is not a possible outcome, the chess theorem states that a game of perfect information has always the same winner. There are nice examples of games in which a very non constructive proof, usually based on some argument by contradiction, shows who is the winner. This is very far from having a winning strategy for the lucky player! One of these funny games is the so called chomp game. A web site where to see it is the following one: http://www.math.ucla.edu/$\tilde{}$tom/Games/chomp.html

2.4 The Chomp Game

The following game is interesting for several reasons. First of all, let us describe it. An $m \times n$ rectangle made by nm squares S_{ij} is given. The first player selects one square, say the square S_{ij}. Then the squares of the form S_{lj}, with $l \leq i$, S_{im}, $m \geq j$ and S_{lm}, $l < i, m > j$, are deleted (see Figures 2.11 and 2.12).

Observe that the analysis of the case $m = n$ is straightforward: a winning strategy for the first player can be easily displayed. As first move, she takes away the square $S_{(n-1)2}$, leaving a symmetric L on the table. Next, the second player must take either a square of the type S_{k1} or S_{nl}. In the first case, the first player takes away the square S_{nk}, in the second case the square S_{l1} (she repeats the move of the second player in a symmetric way with respect to the diagonal). Clearly, this is a winning strategy. But what about if $m \neq n$? Here things are more complicated. However, the following theorem holds.

Theorem 2.4.1 *In the above game, there exists a winning strategy for the first player.*

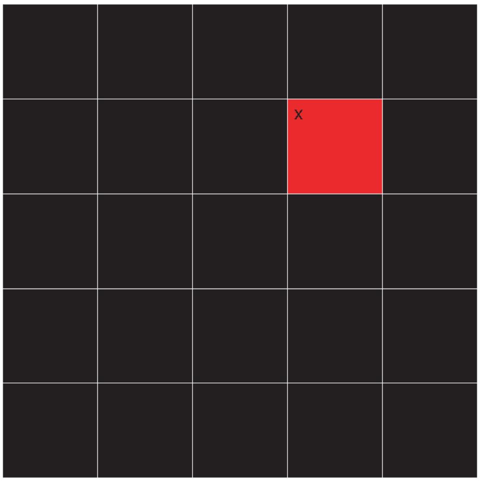

Fig. 2.11. The first player removes the red square

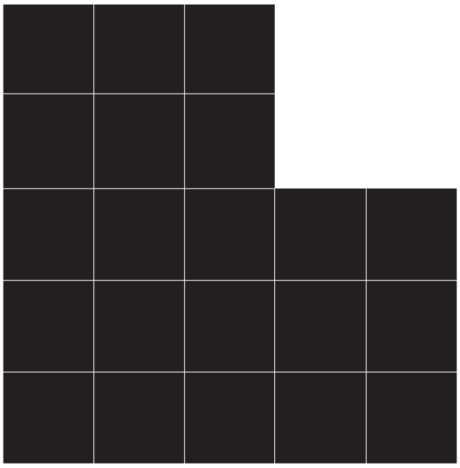

Fig. 2.12. Now it is up to the second player to move

Proof. From the theorem of Zermelo, and since a tie is not allowed in the game, one of the players must have a winning strategy. Suppose it is the second one. Then, he must have a winning move against the move of the first player of taking away the square S_{1n}. Say that the winning move requires to take away the square S_{ij}, with $(i, j) \neq (1, n)$. But then in such a case player one takes away the square S_{ij} at her first move, and this contradiction shows ends the proof. ■

2.5 Combinatorial games

In this section we shall consider a particular family of games of perfect information. Thus the backward induction can apply but, as already observed, more efficient methods are welcome, since the construction of the tree is quite often a too long task. Moreover, as we shall see, this type of games can be handled in full generality, since

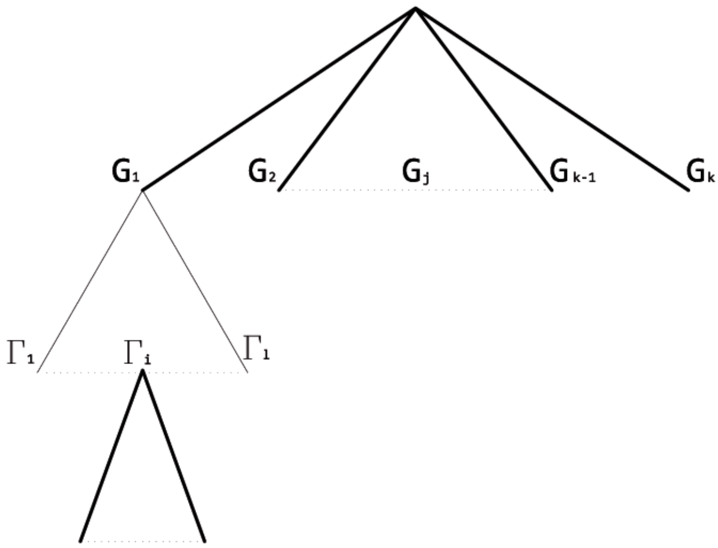

Fig. 2.13. Each of the games Γ_i is isomorphic to some game G_j, but with the players interchanged.

every move in general produces a new game of the same type. Examples below will clarify this point.

Definition 2.5.1 *A* combinatorial game is :

- *a finite game with two players;*
- *with perfect information;*
- *with no chance moves;*
- *there is no possibility of tie;*
- *in the* normal *case the last player to move wins, the opposite in the* misère *case.*

A simple example of this type of games is when the initial condition is to have a given number of piles with some cards in each pile, some rule to taking cards from the piles, till the clearance of the table. In the normal case, the winner is the player clearing the table. Clearly, Zermelo's theorem applies to this class of games; furthermore, there is a winner, since the tie is not admitted. We now see that the computational complexity to analyze in general extensive form games can be here avoided, especially for the so called *impartial* games, i.e. those games where the moves for the players are the same. Observe that chess and checkers are *not* impartial games. Actually, the theory of impartial games was born before Zermelo's theorem, since the solution is given by brute force (thus an existence result is not needed). Let us thus concentrate on impartial games.

2.6 Impartial games

The fundamental idea in this setting is to distinguish among different *positions* in the game, called respectively **P**-positions (also said loosing positions) and **N**-positions

(also said winning positions). Due to the symmetry of the players, what really matters in these games is the state of the game, and not who is called to move. The idea is that the player finding herself in a **N**-position will win: N stands for (*next player to move*) to win. Thus if the game starts with a **N** position it is the first player to win.
On the contrary, if the *previous player* brings the game to a position allowing him to win, then such a state is a **P**-position:thus starting by a **P**-position means that the second wins.

Thus it must be that:

- from a **P**-position it is possible to move *only* to a **N**-position;
- from a **N**-position there must be the possibility to go to a **P**-position.

How is it possible to determine **P** and **N** positions? Needless to say, by backward induction, as follows.

- Terminal nodes are **P**-positions;
- nodes having branches leading to (at least) a **P**-position are **N**-positions;
- nodes having branches leading only to **N**-positions are **P**-positions.

The simplest, and one of the famous, example is probably the following.

2.6.1 The Nim game

In this game, there are N piles of objects (say chips), and one move consists in choosing a pile and take away some chips, at least one (of course!).
In this case a position can be identified by a N-tuple of natural numbers $(n_1, \ldots, n_N)$, where n_i is the number of chips in the i-th pile, $i = 1, \ldots, N$.
The only terminal state is then $(0, \ldots, 0)$, a **P**-position.
Let us start by seeing simple examples. In the case there is only one pile, the game is trivial: every initial position is **N**. Very easy is also the case when there are two piles. In this case the only **P**-positions are those when the two piles have the same number of chips. When a player faces a similar situation, then he is obliged to break it, and the subsequent player can go back to the same number in the two piles.

More interesting the analysis when the piles are more than two. Let us begin with a simple example.

Example 2.6.1 Three piles: initial position $(1, m, m)$, $m \in \mathbb{N}\backslash\{0\}$. In this case, we easily see that $(1, m, m)$ is a **N**-position, since it is enough to go to position $(0, m, m)$. More generally, consider the case:

$$(\underbrace{1, \ldots, 1}_{k}, m, m),$$

where there are k piles made by only one chip, $k \in \{2, 3, 4, 5, \ldots\}$. In this case the general answer is more involved, but we shall see a theorem such that as an immediate corollary we get that the initial position is **P** if and only if k is even.

Example 2.6.2 Let $k \in \mathbb{N}\backslash\{0\}$, $k \neq m$ and consider the initial position: $(1, m, k)$. Before giving a complete answer, let us see some example. $(1, 2, 3)$ is a **P**-position, since the possible moves lead to:

1. *A pile without chips* $\{(0, 2, 3), (1, 0, 3), (1, 2, 0)\}$. In such a case there are only two (nonempty) piles, of different dimension, representing a **N**-position
2. *Two piles with the same number of chips* $\{(1, 1, 3), (1, 2, 2), (1, 2, 1)\}$. These are **N**-positions, since it is enough to eliminate the remaining pile leading to the case of two piles with same number of chips, i.e. to a **P**-position.

In both cases from $(1, 2, 3)$ the player moving needs to go to a **N**-position, whence $(1, 2, 3)$ is a **P**-position.
On the other hand, $(1, 2, 4)$ is a **N**-position, since it is enough to take a chip from the third pile, getting $(1, 2, 3)$, a **P**-position.
Thus the following are **N**-positions: either $(1, 2, k)$, with $k \in \{4, 5, 6, 7, 8, \ldots\}$, or $(1, m, 3)$ with $m \in \{3, 4, 5, 6, \ldots\}$.

Example 2.6.3 A case with four piles. Consider the following position: $(1, 1, 2, 3)$. The possible situations, at te first move:

1. $(0, 1, 2, 3)$. This is a **P**-position -see the example above
2. $(1, 1, 1, 3)$. This is a **N**-position, since by taking two chips from the last chip we get: $(1, 1, 1, 1)$: a **P**-position
3. $(1, 1, 0, 3)$. This is a **N**-position (take all chips from the last pile)
4. $(1, 1, 2, 2)$. This is a **P**-position, since from this we can get either: $(0, 1, 2, 2)$, a **N**-position, or $(1, 1, 1, 2)$, a **N** (see the next step), or else $(1, 1, 0, 2)$, another **N**-position (take the two chips from the last pile)
5. $(1, 1, 2, 1)$. A **N**-position, since you can get: $(1, 1, 1, 1)$, a **P**-position
6. $(1, 1, 2, 0)$. A **N**-position, (take the two chips from the last pile).

Since it is possible to go to a **P**-position, $(1, 1, 2, 3)$ is a **N**-position.
More precisely, it is possible to go in three **P**-positions: $(0, 1, 2, 3)$, $(1, 0, 2, 3)$ and $(1, 1, 2, 2)$.
The fact that the following **P**-positions are in an *odd* number is not a feature of this example, but a general fact, as we shall see.

Nimbers

Let us see the general way to tackle the problem of solving the Nim game. The smart idea is to define on the natural numbers a new operation, making it an abelian group, and giving the way to solve the game.

Define a new operation $+_{\mathcal{N}}$ on the natural numbers as follows. For $n, m \in \mathbb{N}$:

1. write n, m in binary base: $n = (n_k \ldots n_0)_2$, $m = (m_k \ldots m_0)_2$
2. form the binary number $(n_k + m_k)_2 \ldots (n_0 + m_0)_2$
3. define $n +_{\mathcal{N}} m = z$, where $(z)_2 = (n_k + m_k)_2 \ldots (n_0 + m_0)_2$.

The elements of $(\mathbb{N}, +_{\mathcal{N}})$ are often called *nimbers*.

Example 2.6.4 1. $2 +_{\mathcal{N}} 4 = 6$: $(2)_2 = 10$, $(4)_2 = 100$, $10 +_2 100 = 110$ and $6 = (110)_2$

2. $2 +_{\mathcal{N}} 3 = 1$. $(2)_2 = 10$, $(3)_2 = 11$, $10 +_2 11 = 01 = 1$, $(1)_2 = 1$
3. $3 +_{\mathcal{N}} 3 = 0$, more generally $n +_{\mathcal{N}} n = 0$.

The structure of $\mathbb{N}$ with the Nim sum is interesting.

Proposition 1. *The Nimbers set in an abelian group.*[8]

Proof. We only observe that the identity element is 0, and the inverse of n is n itself. Associativity and commutativity of the operation are easy matters. ■

Note that in any group the cancellation law holds: $n_1 +_{\mathcal{N}} n_2 = n_1 +_{\mathcal{N}} n_3$ implies $n_2 = n_3$.

We already remarked that when having two piles of chips, a pair (n, m) is a **P**-position if and only if $n = m$. This condition reads, within nimbers, that the sum $n +_{\mathcal{N}} m = 0$. Also, we observed before that $(1, m, m)$ is a **N**-position. And $1 +_{\mathcal{N}} m +_{\mathcal{N}} m = 1 +_{\mathcal{N}} 0 = 1 \neq 0$. It turns out that this fact can be fully generalized: this is the content of the following theorem.

Theorem 2.6.1 (Bouton) *A* $(n_1, n_2, \dots, n_N)$ *position in the Nim game is a* **P**-*position if and only if* $n_1 +_{\mathcal{N}} n_2 +_{\mathcal{N}} \dots +_{\mathcal{N}} n_N = 0$.

Proof. Let $\mathfrak{P}$ be the set of positions with null Nim sum and $\mathfrak{N}$ the set of positions with positive Nim sum.

- *Terminal states belong to* $\mathfrak{P}$. This is obvious;
- *Positions in* $\mathfrak{P}$ *go only to positions in* $\mathfrak{N}$. Let $n_1 +_{\mathcal{N}} n_2 +_{\mathcal{N}} \dots +_{\mathcal{N}} n_N = 0$ and suppose, f.i. to take some chips from n_1, to get $n_1' < n_1$.
  Suppose $n_1' +_{\mathcal{N}} n_2 +_{\mathcal{N}} \dots +_{\mathcal{N}} n_N = 0 = n_1 +_{\mathcal{N}} n_2 +_{\mathcal{N}} \dots +_{\mathcal{N}} n_N$; from the cancellation law we would get $n_1' = n_1$. This is impossible;
- *Positions in* $\mathfrak{N}$ *can go to a position in* $\mathfrak{P}$. Let $z \in \mathfrak{N}$; then $z := n_1 +_{\mathcal{N}} n_2 +_{\mathcal{N}} \dots +_{\mathcal{N}} n_N \neq 0$. Write z in binary base. Since $z \neq 0$, there is at least one 1 in its expansion in binary base. Consider the left most 1 in the digit expansion of z and select a pile having a corresponding 1 in its expansion. Put zero there and change the digits on the right in order to make the final sum to be zero. Observe that doing this is possible since the nimber so obtained is smaller than the previous one; in other words, some chip has been taken from the pile.

■

Example 2.6.5 Let us go back to the situation:

$$(\underbrace{1, \dots, 1}_{k}, m, m),$$

$k \in \{2, 3, 4, 5, \dots\}$. If k is even, the 1's simplify, then

[8] A nonempty set A with an operation $\cdot$ on it is called a *group* provided:

1. $\cdot$ is associative
2. there is an element (which will result unique, and called *identity*) e such that $a \cdot e = e \cdot a = a$ for all $a \in A$
3. for every $a \in A$ there is $b \in A$ such that $a \cdot b = b \cdot a = e$. Such an element is unique and called *inverse* of a.

If $a \cdot b = b \cdot a$ for all $a, b \in A$ the group is called *abelian*.

$$1 +_{\mathcal{N}} \ldots +_{\mathcal{N}} 1 +_{\mathcal{N}} m +_{\mathcal{N}} m = 0 +_{\mathcal{N}} m +_{\mathcal{N}} m = 0.$$

Instead, if k is odd:

$$1 +_{\mathcal{N}} \ldots +_{\mathcal{N}} 1 +_{\mathcal{N}} m +_{\mathcal{N}} m = 1 +_{\mathcal{N}} m +_{\mathcal{N}} m = 1.$$

Hence if k is even the position is a **P**-position. Let us see, for k odd, how to go to a **P**-position. The simple idea is to eliminate one 1. Let us do it by following the proof of the theorem:

$$z := 1 +_{\mathcal{N}} \ldots +_{\mathcal{N}} 1 +_{\mathcal{N}} m +_{\mathcal{N}} m = 1 +_{\mathcal{N}} m +_{\mathcal{N}} m = 1,$$

whence $(z)_2 = 1$. Then we need to take chips from a pile having 1 in the last digit on the right. It is possible to do it from one of the first k. Observe that it could possibly be the case also for the two piles having m elements. Summarizing, we can work on either k or $k + 2$ piles.

Example 2.6.6 Initial position $(1, 2, 4, 1)$. Writing the nimbers for the sum:

$$\begin{array}{c} 0\,0\,1 \\ 0\,1\,0 \\ 1\,0\,0 \\ \underline{0\,0\,1} \\ \\ 1\,1\,0 \end{array}$$

and thus

$$n_1 +_{\mathcal{N}} n_2 +_{\mathcal{N}} n_3 +_{\mathcal{N}} n_4 = 6.$$

So that $(1, 2, 4, 1)$ is a **N** position, the first one to move wins. We have $(6)_2 = 110$. In order to move to a **P**-position, from the argument of the theorem one has to move to:

$$\begin{array}{c} 0\,0\,1 \\ 0\,1\,0 \\ 0\,1\,0 \\ \underline{0\,0\,1} \\ \\ 0\,0\,0 \end{array}$$

The situation is now the following:$(1, 2, 1, 2)$. An easy argument shows that the only possible situations[9] after the move of the second player are the following: $(0, 2, 1, 2)$,$(1, 1, 1, 2)$, $(1, 0, 1, 2)$. Now it is very easy to see how to proceed.

From the proof of the theorem, it is easy to get the following result:

Corollary 2.6.1 *From a* **N***-position there are an odd number of moves going to a* **P***-position.*

[9] Up to rearranging piles.

Proof. Easy. ■

It is possible to generalize the Nim game, allowing the players to take a positive number of chips from k piles, with $1 \leqslant k \leqslant N$. If $k = 1$ it is the usual Nim, if $k = N$ the game is totally silly.
The theorem of Bouton can be generalized to get:

Theorem 2.6.2 *A position $(n_1, \dots, n_N)$ in the Nim$_k$ game is a **P**-position if and only if the sum* modulo $k+1$ *as done before of the numbers $n_1, \dots, n_N$ written in binary base is null.*

Proof. Exercise.

Example 2.6.7 Consider three piles, suppose $k = 2$ and initial position $(1, 2, 4)$.

$$\begin{array}{c} 0\,0\,1 \\ 0\,1\,0 \\ \underline{1\,0\,0} \\ \\ 1\,1\,1 \end{array}$$

The first player wins. Let us understand haw she can lead to situation to a **P**-situation, i.e. with sum zero.
The only remark to do is that where one finds 1 in the sum, having three piles this means that in the other two piles there is 0. In the most right digit of the two other piles we need to find, after the moves, 1 (in order to get $3 = 0 \bmod 3$). As far as the other digits of the two piles are concerned, we can put 0 where we have 1, in order to get final sum zero. Thus:

$$\begin{array}{c} 0\,0\,1 \\ 0\,0\,1 \\ \underline{0\,0\,1} \\ \\ 0\,0\,0 \end{array}$$

The correct move is to leave $(1, 1, 1)$ on the table.

There are possible generalizations to the previous theorems. In general it is possible to make a complete theory for impartial games. We do not pursue this subject here. We just add a couple of examples.

Example 2.6.8 Suppose we have a pile with n chips, and the players can take at most k chips from it. Who takes the last one (possible) wins. As before , the idea is to partition the positions in **P**- and **N**-positions. In this example, the situation is easy. 0 is a **P**-position, while $1, \dots k$ are clearly **N**-positions. What about $k + 1$? Clearly, we can only leave on the chips a number of piles less or equal to k. In other words, $k + 1$ is a **P**-position. It does not take much time to realize then that the first player always wins, unless $n = 0 \bmod k + 1$. Suppose now to face the same problem, but the players can take a fixed amount of piles, say $k_1, \dots, k_j$. We can proceed with similar techniques. For instance, suppose it is possible to take off two three or five piles. 0 and

1 are **P**-positions. 2 is a **N**-position (go to 0), 3 is a **N**-position (go either to 0 or 1), 4 is a **N**-position (go to 1), 5 is a **N**-position (go to 0),6 is a **N**-position (go to 1), 7 is a **P**-position (going to 2,4,5) .... Thus a position is a **P**-position if and only if $n \in \{0, 1\}$ mod 7. A further generalization can be provided by considering, for instance, k piles with $n_1, \dots, n_k$ chips respectively, and to establish a rule for taking off piles from each pile. The player can choose the pile, and how many chips take from the pile (following the corresponding rule). An interesting theorem shows actually how to reduce this type of problem to a Nim problem, via the so called *Sprague-Grundy* function.

2.7 Conclusions

We saw in this chapter a relevant result related to the specific class of the perfect information games: the so called Zermelo theorem. We also saw how the result can be translated in different situations. It is clear that the result, as applied to the game of chess, to the rectangles game and to the combinatorial games, looks different. In is worth, in this sense, to make a distinction among the different situation. We can speak about a *very weak solution* when we have no more information on the game than in the case of chess: the game is determined, but it is not known how. A *weak solution* instead is the situation when, like in the rectangles case, we can establish the outcome of the game, without providing (in general, not for particular cases, where a winning strategy can be found) though the winning strategy is unknown. Finally, we speak about *solution* when we are able not only to say who is the winner (or if a tie is the result of the game), but also to exhibit optimal strategies, as in the combinatorial games.

Let us conclude with some curiosity. First of all, rather recently a weak solution of the checkers has been announced: the outcome should be the tie. Moreover, both in case of chess and checkers it is known the exact number of nodes of the tree when limited to length eleven (1 correspond to length zero and represents the root of the tree): Chess [1, 20, 400, 8902, 197281, 4865609, 119060324, 3195901860, 84998978956, 2439530234167, 69352859712417, 2097651003696806] and Checkers [1, 7, 49, 302, 1469, 7361, 36768, 179740, 845931, 3963680, 18391564, 85242128]. It appears very immediately how much the chess is more complicated. It is out of question to know the exact number of possible games in the game of chess but Shannon (the father of the information theory) guessed that this number, assuming a standard length of 40 moves per game, should be 10^{100}, much more than the estimated number of the atoms of the universe, which is 10^{80}.

2.8 Exercises

Exercise 2.8.1 Describe a game in extensive form with 2 players, where player I has 8 strategies and player II 4. Write another one of length 3 and one of length 4. No player can make two consecutive moves.

Solution We represent such a game in Figure 2.14. Strategies for I are AGI, AGL, AHI, AHL, BGI, BGL, BHI, BHL. Strategies for II are CE, CF, DE, DF.

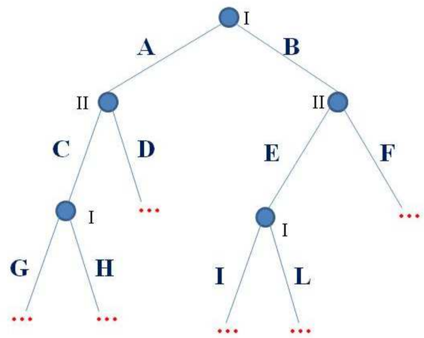

Fig. 2.14. Exercise 2.8.1

Exercise 2.8.2 Describe a game in extensive form with 2 players, where player I has 8 strategies and player II 17.

Solution We represent such a game in Figure 2.15.

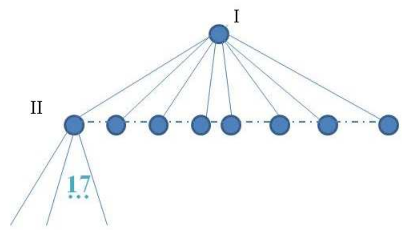

Fig. 2.15. Exercise 2.8.2

Exercise 2.8.3 Write the strategic form of the game in Figure 2.16.

Solution Both the players have 4 strategies. The strategic form of the game is given by the following bimatrix:

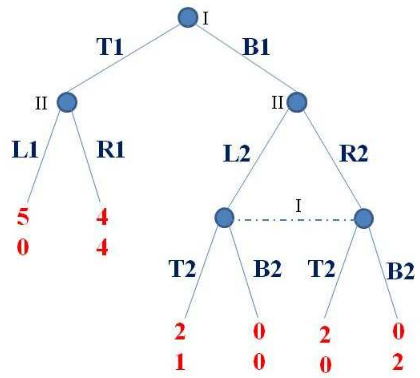

Fig. 2.16. Exercise 2.8.3

$$\begin{array}{c} \\ T_1T_2 \\ T_1B_2 \\ B_1T_2 \\ B_1B_2 \end{array} \begin{array}{c} \quad L_1L_2 \; L_1R_2 \; R_1L_2 \; R_1R_2 \\ \begin{pmatrix} (5,0) & (5,0) & (4,4) & (4,4) \\ (5,0) & (5,0) & (4,4) & (4,4) \\ (2,1) & (2,0) & (2,1) & (2,0) \\ (0,0) & (0,2) & (0,0) & (0,2) \end{pmatrix} \end{array}$$

Exercise 2.8.4 A mixed strategy is a probability distribution over the set of pure strategies. A behavioral strategy is a probability distribution over the branches (moves) coming out from each information set. Find how many pure, mixed and behavioral strategies there are in the following game.
Given an ordinary deck with cards labelled from 1 to 13. Two cards are taken randomly. One is shown to you. If you do not want to bet, you pay 2 to the second player. Otherwise, you get from her the difference between the number in your card and the number in the other card. Who wins and how much?

Solution There are $2^{13} = 8192$ pure strategies (for every number between 1 and 13, the player can decide to bet or not), the 8192-th simplex for mixed strategies, the unit 13-th dimensional cube for behavioral strategies. Suppose a 13 is shown to you. Then you get:

$$(12 + 11 + \cdots + 1) \times \frac{4}{51}.$$

So that the expected values are:

$$\{-\frac{312}{51}, -\frac{260}{51}, -\frac{208}{51}, -\frac{156}{51}, -\frac{104}{51}, -\frac{52}{51}, 0, \frac{52}{51}, \frac{104}{51}, \frac{156}{51}, \frac{208}{51}, \frac{260}{51}, \frac{312}{51}\}.$$

It is convenient to play only if the expectation > -2. Since $-\frac{104}{51} < -2 < -\frac{52}{51}$, the unique optimal behavioral strategy is $(0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1)$ with total expectation $\frac{530}{13 \cdot 51} \approx 0.8$.

Exercise 2.8.5 The following is not a game. Why? An absent minded driver must come back home from work. He knows he must turn right at the first round point and left at the second one. Unfortunately, quite often he forgets if he already passed the first round point! What would he get by playing pure or mixed strategies here? What should he do without game theory?

Solution He has two strategies, left, right. According to the definition of strategy he gets 0 from both. So, he gets the same by mixing. But he can get better if he toss a coin any time he arrives to a round point (using a behavioral strategy).

Exercise 2.8.6 Represent the Voting Game (Example 1.3.2) in strategic form.

Solution We can represent the game in the following way, writing down the outcomes of the game. Player III chooses the matrix, I the row and II the column.

A

	A	B	C
A	A	A	A
B	A	B	B
C	A	C	C

B

	A	B	C
A	A	B	A
B	B	B	B
C	C	B	C

C

	A	B	C
A	A	A	C
B	B	B	C
C	C	C	C

Supposing they have utility 3 if the outcome is their preferred one, 2 if it is the second and 1 if it is the worst one for them, we can write the game in strategic form in the following way

$$\begin{pmatrix} (3,1,2) & (3,1,2) & (3,1,2) \\ (3,1,2) & (2,3,1) & (2,3,1) \\ (3,1,2) & (1,2,3) & (1,2,3) \end{pmatrix} \begin{pmatrix} (3,1,2) & (2,3,1) & (3,1,2) \\ (2,3,1) & (2,3,1) & (2,3,1) \\ (1,2,3) & (2,3,1) & (1,2,3) \end{pmatrix} \begin{pmatrix} (3,1,2) & (3,1,2) & (1,2,3) \\ (2,3,1) & (2,3,1)) & (1,2,3) \\ (1,2,3) & (1,2,3) & (1,2,3) \end{pmatrix}$$

Exercise 2.8.7 Represent the game of the three politicians in the Example 2.1.1 in strategic form.

Solution We have the following matrices when player I chooses a

	$gjln$	$gjlo$	$gjmn$	$gjmo$	$gkln$	$gklo$	$gkmn$	$gkmo$	$hjln$	$hjlo$	$hjmn$	$hjmo$	$hkln$	$hklo$	$hkmn$	$hkmo$
ce	β,β,β	β,β,β	β,β,β	β,β,β	β,β,β	β,β,β	β,β,β	β,β,β	β,β,α	$\beta,\beta,\alpha)$	β,β,α	β,β,α	β,β,α	β,β,α	β,β,α	β,β,α
cf	β,β,β	β,β,β	β,β,β	β,β,β	β,β,β	β,β,β	β,β,β	β,β,β	β,β,α	β,β,α	β,β,α	β,β,α	β,β,α	β,β,α	β,β,α	β,β,α
de	β,α,β	β,α,β	β,α,β	β,α,β	δ,γ,γ	δ,γ,γ	δ,γ,γ	δ,γ,γ	β,α,β	β,α,β	β,α,β	β,α,β	δ,γ,γ	δ,γ,γ	δ,γ,γ	δ,γ,γ
df	β,α,β	β,α,β	β,α,β	β,α,β	δ,γ,γ	δ,γ,γ	δ,γ,γ	δ,γ,γ	β,α,β	β,α,β	β,α,β	β,α,β	δ,γ,γ	δ,γ,γ	δ,γ,γ	δ,γ,γ

and when player I chooses b

	$gjln$	$gjlo$	$gjmn$	$gjmo$	$gkln$	$gklo$	$gkmn$	$gkmo$	$hjln$	$hjlo$	$hjmn$	$hjmo$	$hkln$	$hklo$	$hkmn$	$hkmo$
ce	α,β,β	α,β,β	γ,γ,γ	γ,γ,γ	α,β,β	α,β,β	γ,γ,γ	γ,γ,γ	α,β,β	$\alpha,\beta,\beta)$	γ,γ,γ	γ,γ,γ	α,β,β	α,β,β	γ,γ,γ	γ,γ,γ
cf	γ,γ,δ	γ,γ,γ	γ,γ,δ	γ,γ,γ	γ,γ,δ	γ,γ,γ	γ,γ,δ	γ,γ,γ	γ,γ,δ	γ,γ,γ	γ,γ,δ	γ,γ,γ	γ,γ,δ	γ,γ,γ	γ,γ,δ	γ,γ,γ
de	α,β,β	α,β,β	γ,γ,γ	γ,γ,γ	α,β,β	α,β,β	γ,γ,γ	γ,γ,γ	α,β,β	$\alpha,\beta,\beta)$	γ,γ,γ	γ,γ,γ	α,β,β	α,β,β	γ,γ,γ	γ,γ,γ
df	γ,γ,δ	γ,γ,γ	γ,γ,δ	γ,γ,γ	γ,γ,δ	γ,γ,γ	γ,γ,δ	γ,γ,γ	γ,γ,δ	γ,γ,γ	γ,γ,δ	γ,γ,γ	γ,γ,δ	γ,γ,γ	γ,γ,δ	γ,γ,γ

Exercise 2.8.8 Represent the Russian Roulette in strategic form.

Solution The strategies of I are shooting (S) or passing (P). The strategies of II are a little bit more complicated, in particular he can decide to shoot both if I shoots or if I does not (S_1S_2), to shoot if I shot and not to if I did not (S_1P_2), not to shoot if I shot and viceversa (P_1S_2), not to shoot in any case (P_1P_2). The game in strategic form is given by the following bimatrix

$$\begin{pmatrix} -\frac{2}{36} & -\frac{2}{36} & \frac{1}{12} & \frac{1}{12} \\ \\ -\frac{1}{12} & 0 & -\frac{1}{12} & 0 \end{pmatrix}$$

Exercise 2.8.9 Two players. A pile of 21 chips. A move consists in removing 1,2, or 3 chips. The player removing the last one wins. Who wins?

Solution The first one wins. He must take one chip at the first stage. Then at the second move he leaves 16, then 12, etc. For a better formalization see the solution of the following exercise.

Exercise 2.8.10 Same game as before, with 79 chips, but now it is possible to take 1,3,4 chips

Solution This type of game can be analyzed in the following way. Call P and N positions the situations in the game, defined recursively:

1. terminal positions are P positions
2. from every N there is at least a move on a P position
3. from every P position, all moves go to an N position.

The idea is the following: in this game, the terminal position is 0. Who is able to arrive to it, wins. Suppose the game starts with a P position. The first is forced to go to a N position. Then the second can go to a P position.... Thus the second player wins if and only if the initial position is a P position. According to the definition,$1, 3, 4$ are N positions since they can be moved to 0. 2 is a P position, since the only possible move is to go to 1. $5, 6$ are in N, since they can be moved in 2. 7 is N, since it can be moved in $3, 4, 6$ which all are P.

$$\overset{P}{0}\ \overset{N}{1}\ \overset{P}{2}\ \overset{N}{3}\ \overset{N}{4}\ \overset{N}{5}\ \overset{N}{6}\qquad \overset{P}{7}\ \overset{N}{8}\ \overset{P}{9}\ \overset{N}{10}\ \overset{N}{11}\ \overset{N}{12}\overset{N}{13}\qquad \dots$$

Since $79 \equiv 2$ (Mod 7), and 2 is P, then the second player wins (she takes away 2 in the first move).

Exercise 2.8.11 Two piles of chips. Two players take, alternatively, either as many chips as they wish from one pile, or the same number from both. The player leaving the table empty wins the play. Show that that the second player wins if and only if the initial configuration is in the set E of the following pairs:

$$E = \{0, 0), (1, 2), (3, 5), (4, 7) \dots\}.$$[10]

But this exactly the form of a typical pair on E. Thus every pair contained in E can be written as $([nt], [nt] + n)$, with $n \geq 0$.

Solution The idea of the proof is similar to the above exercise. One must show that, starting from a pair inside E, all available moves result in a pair not in E. And, starting from a pair not in E, it is possible to enter again in E. Thus, at the first move, the first player go out of E, the second is able to get in again etc. It is very easy to show that starting from a pair in E you cannot select another pair in E. For, if you select from one pile, you do not change one of the elements of the pair, but the other one is changed. Thus you are not any more in E, since every number appears only once in a pair in E. On the other hand, if you remove from both, you maintain

[10] The following observation, though not essential in proving the result, is useful. The set E contains all positive natural numbers, that appear only once. This depends on the fact that, if $a > 1, b > 1$ are irrational numbers such that $\frac{1}{a} + \frac{1}{b} = 1$, it can be shown that every natural number can be written either as $[na]$ or $[nb]$. Setting $t = \frac{1+\sqrt{5}}{2}$, since $\frac{1}{t} + \frac{1}{t+1} = 1$, then every natural number can be written (in a unique way) either as $[nt]$ or as $[nt] + n$.

the same difference between the two numbers of the pair, and so you go out from E. Now the interesting part, describing a winning strategy. Suppose the pair is of the form (a, b), with $a < b$. Suppose $a = [nt] + n$, for some n. So take away from chips from b in order to leave on the table the quantity $[nt]$. If instead $a = [nt]$, for some n, two possible cases occur:

1. $b > [nt] + n$: in this case leave $[nt] + n$ chips in the second pile
2. $b < [nt] + n$. Take then the quantity $a - [(b - a)t]$ chips from both piles. You get $[(b - a)t], [(b - a)t] + (b - a)$.

Exercise 2.8.12 More questions related to the Exercise above:

1. Show that the winning strategy is not unique in general
2. In how many moves can the first player win starting from $(7, 9)$?

Solution

1. Consider for instance $(10, 12)$. It can go either to $(6, 10)$ or to $(3, 5)$
2. From $(7, 9)$ to $(3, 5)$ (better than $(4, 7)$). Possibilities:

$$(3, 4), (3, 3), (3, 2), (3, 1), (3, 0), (4, 2)(2, 0).$$

$(3, 3), (3, 0), (2, 0)$ go to a final situation. The other ones have best answer $(1, 2)$. Game over in the next move by player one.

Exercise 2.8.13 Proof of Zermelo's Theorem for a combinatorial game without chance and draw.

Solution An optimal strategy for the first player can be written as:
$\exists$ a choice of I s.t. $\forall$ choice of II $\exists$ a choice of I... I wins.
If we deny the previous sentence, we write:
$\forall$ choice of I $\exists$ a choice of I s.t. $\forall$ choice of I... I does not win.
I does not win is the same that II wins, as we do not admit a draw. Then the last sentence is exactly the definition of the fact that II has a winning strategy.
As one of the two sentences has to be true, if I does not win it is true the opposite, i.e. that II wins.

Exercise 2.8.14 Consider the misère version of the take-away game 2.8.9, where the last player to move loses. The object is to force your opponent to take the last chip. Analyze this game. What are the target positions (P-positions)?

Solution With a misère play rule all terminal positions are N-positions:

N	P	N	N	N	P	N	N	N	P	N	N	N	P	N	N	N	P	N	N	N	P
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21

Player I is going to loose, whatever he does player II will move to a P-position, forcing him to an N-position.

Exercise 2.8.15 Consider the following Take-Away Game:

- Suppose in a game with a pile containing a large number of chips, you can remove any number from 1 to 6 chips at each turn. What is the winning strategy? What are the P-positions?

- If there are initially 31 chips in the pile, what is your winning move, if any?

Solution Showing the N and the P-positions

$$\begin{array}{cccccccccccccc} P & N & N & N & N & N & N & P & N & N & N & N & ... \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & ... \end{array}$$

as $31 = 3\text{mod}(7)$, 31 is an N-position and player II is going to win taking 3 chips from the pile.

Exercise 2.8.16 From a deck of cards, take the Ace, 2, 3, 4, 5, and 6 of each suit. These 24 cards are laid out face up on a table. The players alternate turning over cards and the sum of the turned over cards is computed as play progresses. Each Ace counts as one. The player who first makes the sum go above 31 loses. It would seem that this is equivalent to the game of the previous exercise played on a pile of 31 chips. But there is a catch. No integer may be chosen more than four times.

- Can the first player win by choosing at the beginning the winning strategy of the previous case (taking away three cards)?
- Find a winning strategy for the first player.

Solution We write the P-positions and the N-positions following the previous scheme, just considering 31 as the terminal position.

$$\begin{array}{ccccccccccccccccc} P & N & N & N & N & N & N & P & N & N & N & N & N & N & P & N & N \\ 31 & 30 & 29 & 28 & 27 & 26 & 25 & 24 & 23 & 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 \end{array}$$

$$\begin{array}{ccccccccccccccc} N & N & N & N & P & N & N & N & N & N & N & P & N & N & N \\ 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{array}$$

No integer may be chosen more than four times. If player I decides to take a 3 as first move in order to go to a P-position, player II will reply taking a 4. Since player I cannot keep choosing 3 as after four moves he is forced to choose a different card. In this way he will loose. The first player can win with optimal playing choosing 5 at the beginning. Player I does not go to an N-position, but in this way player II, if decides to go to a P-position, has to take another 5. But then player I will play a 2 in order to force player II to take a 5 again to go to a P-position. If player I keeps choosing a 2, player II cannot take as many 5 as he needs and he is forced to take a different card, going to an N-position and allowing player I to go back to a P-position and to win the game. On the other hand, if player II decides to go to a N-position, it is possible to see that then player I, going to a P-position, in the subsequent steps will be able to maintain this choice for ever, since he never needs a card already selected four times.

Exercise 2.8.17 Take the Game of Nim with initial position $(13, 13, 8)$. Is this a P-position? If not, what is a winning move?

Solution $(13, 12, 8)$ is a P-position if and only if the nim-sum is equal to zero.

$$13 = 1101_2$$

$$12 = 1100_2$$

$$8 = 1000_2$$

and the nim-sum is

$$\begin{array}{l} 1101 + \\ 1100 + \\ 1000 = \\ \hline 1101 \end{array}$$

The starting position is a P-position. A winning move is given by bringing to zero the nim-sum, choosing to remove 9 chips from the first pile or 7 from the second or 7 from the third one.

Exercise 2.8.18 Turning Turtles A horizontal line of n coins is laid out randomly with some coins showing heads and some tails. A move consists of turning over one of the coins from heads to tails, and in addition, if desired, turning over one other coin to the left of it (from heads to tails or tails to heads). For example consider the sequence of $n = 13$ coins:

$$\begin{array}{ccccccccccccc} T & H & T & T & H & T & T & T & H & H & T & H & T \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array}$$

One possible move in this position is to turn the coin in place 9 from heads to tails, and also the coin in place 4 from tails to heads.

- Show that this game is just nim in disguise if an H in place n is taken to represent a nim pile of n chips.
- Assuming the previous point to be true, find a winning move in the above position.

Solution This game is a nim game, where every T represents an empty pile and every H in n position represents a pile with n chips. We have

$$\begin{array}{ccccccccccccc} T & H & T & T & H & T & T & T & H & H & T & H & T \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array}$$

i.e.

$$(0, 2, 0, 0, 5, 0, 0, 0, 9, 10, 0, 12, 0)$$

Turning one coin from head to tail is equal to removing all the chips from one pile. When we want to remove a number m of chips from a pile of $n > m$ chips, this is equal to turning the coin in position n and the one in position $n - m$. As it is a nim game we evaluate the nim-sum

$$2 = 10_2$$

$$5 = 101_2$$

$$9 = 1001_2$$

$$10 = 1010_2$$

$$12 = 1100_2$$

$$\begin{array}{r} 1100 + \\ 1010 + \\ 1001 + \\ 101 + \\ 10 = \\ \hline 1000 \end{array}$$

The nim-sum is not equal to zero, i.e. this is an N-position. A winning move is given by bringing the nim-sum to zero, removing 8 chips from the pile in position 12, in our game this is equal to turning the coins in position 12 and 8.

Exercise 2.8.19 What happens when we play nim under the misère play rule? Can we still find who wins from an arbitrary position, and give a simple winning strategy? Start with the situation $(5, 3, 7)$ and generalize the result.

Solution Here is Bouton's method for playing misère nim optimally. Play it as you would play nim under the normal play rule as long as there are at least two cheaps of size greater than one. When your opponent finally moves so that there is exactly one pile of size greater than one, reduce that pile to zero or one, whichever leaves an odd number of piles of size one remaining. This works because your optimal play in nim never requires you to leave exactly one pile of size greater than one (the nim sum must be zero), and your opponent cannot move from two piles of size greater than one to no piles greater than one. So eventually the game drops into a position with exactly one pile greater than one and it must be your turn to move.

Exercise 2.8.20 We consider the Nim_2 game given by the initial position $(2, 3, 1, 1)$. Who wins?

Solution $(2, 3, 1, 1)$ is a P-position if and only if the nim-sum $mod(3)$ is equal to zero.

$$2 = 10_2$$

$$3 = 11_2$$

$$1 = 1_2$$

and the nim-sum is

$$\begin{array}{l} 10\ + \\ 11\ + \\ 01\ + \\ 01\ = \\ \hline 20 \end{array}$$

The starting position is an N-position. The first player wins choosing to remove, for example, 2 chips from the first and from the second pile.

Exercise 2.8.21 We consider a Nim game with 3 piles. We sort the number of chips in each pile between 0 and 7. Evaluate the probability of obtaining a P-position (in this situation player II is going to win). Generalize to an n chips situation.

Solution A number between 0 and 7, in binary, can be written with 3 digits. Writing a matrix with the three number written in three digits we have

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{array}$$

All possible combinations of numbers are 2^9 (every digit can be a 0 or a 1). In particular we have a P-position when in every column we have a number of 1s equal to zero or two, i.e. in every column we can have 0 0 0, 1 1 0, 1 0 1 or 0 1 1, four

combinations times three columns, i.e. $4^3 = 64$ cases. The probability to obtain a P-position is $\frac{4^3}{2^9}$.
When the number of piles is n, the result has the generalized formula

$$\frac{\left[\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k}\right]^3}{2^{3n}}$$

and moreover when we assume to sort numbers which can be written with l digits

$$\frac{\left[\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k}\right]^l}{2^{ln}}$$

Is it easy to prove that this formula is always equal to $\frac{1}{2^l}$. We could have shown the result in a easier way, just assuming that all the number with l digits in binary base are 2^l and they have all the same probability to be obtained from a nim-sum. The only one which guarantees a P-position is the vector of all zero, i.e. we have a P-position every 2^l combinations.

Exercise 2.8.22 Imagine in your mind a rather big matrix, representing the chess in strategic form, and describe what the matrix should contain in each of the previous three alternatives. Think also to which alternative(s) the theorem excludes in these situations.

Exercise 2.8.23 Consider the infinite case of the Chomp game. Prove that the winner is the first player, unless the rectangle in $2 \times \infty$, where the second player has a winning strategy.

Exercise 2.8.24 Stackelberg duopoly model. Two firms produce quantities q_1 and q_2 respectively, the market requires a maximal amount a of good. Utility $u_i(q_1, q_2) = \max\{q_i(a - (q_1 + q_2) - c), 0\}$, where $a - (q_1 + q_2)$ is the unitary price of the good, and c is the unitary cost to produce it. First player announces the quantity $\bar{q}_1$ will produce. Next the second player makes his move. Find the outcome of the game.

Exercise 2.8.25 Explain what happens in the following four games, enumerate their strategies and construct the corresponding (bi)matrices.

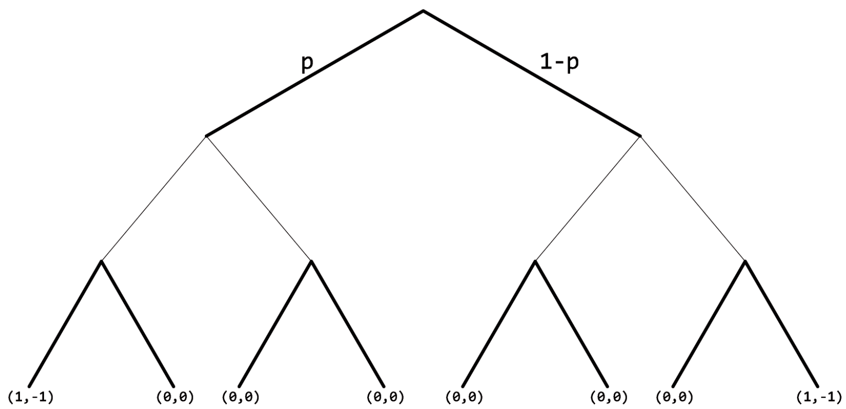

Fig. 2.17. Exercise 2.8.25

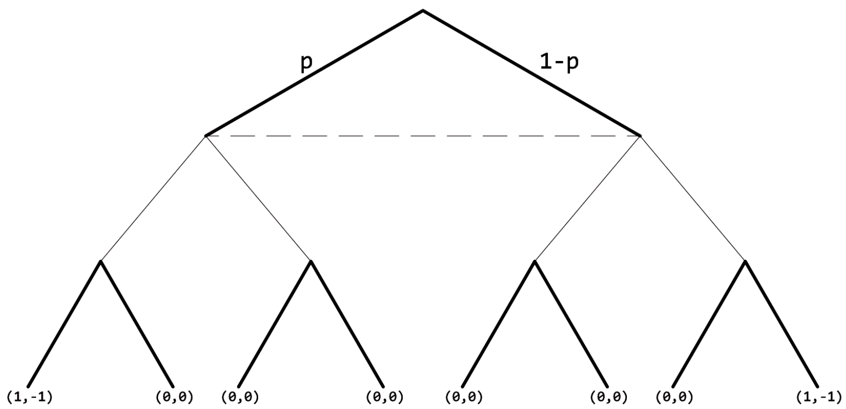

Fig. 2.18. Exercise 2.8.25

Exercise 2.8.26 Consider the extended nim game with the possibility to take off from at most two piles. Verify that $(5, 3, 3, 7)$ is a **N**-position.

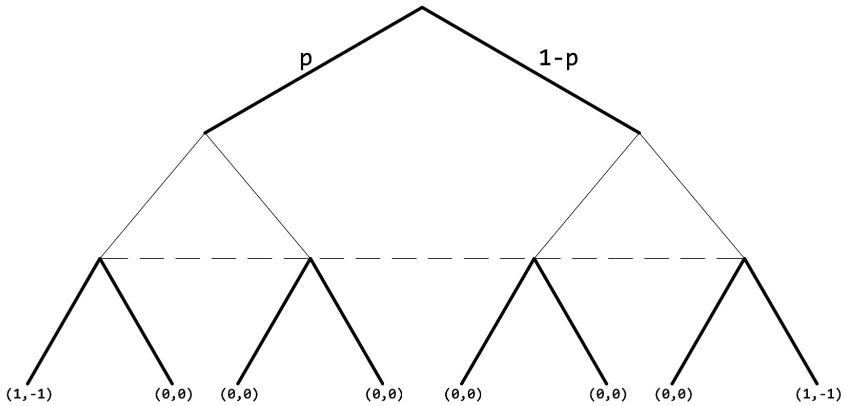

Fig. 2.19. Exercise 2.8.25

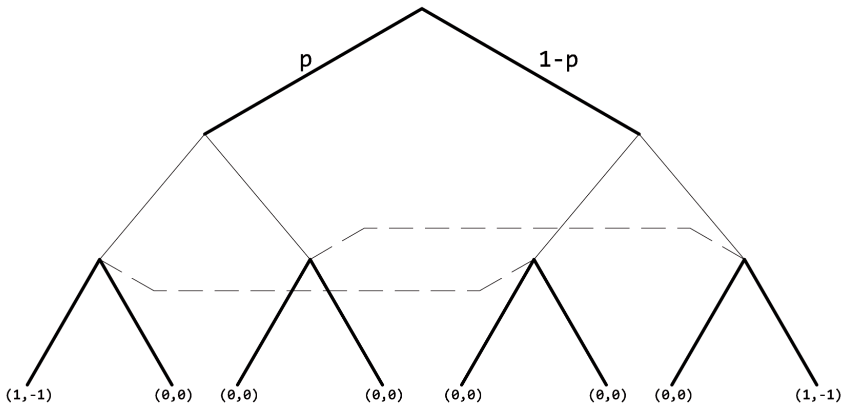

Fig. 2.20. Exercise 2.8.25

3

Zero sum games

In this chapter we start by analyzing the games in strategic form, in order to find novel ideas to define a rational behaviour for the players. So far, in order to find out what a solution could be, we have at first introduced a rationality criterion, that essentially suggests to reject dominated strategies: let $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$ be a game. A strategy $x \in X$ is dominated (weakly) by a strategy $z \in X$ for player one if $f(x,y) < f(z,y)$ for all $y \in Y$ ($f(x,y) \leq f(z,y)$ for all $y \in Y$). But this usually allows eliminating very few possible outcomes of the game. Furthermore, in the case of extensive games with perfect information, the backward induction allows us introducing an efficient rationality property, which provides the correct outcome(s) of the game. However, it is quite clear that most games require simultaneous moves to the players, and this of course makes the game impossible to be described as a game of perfect recall. Thus, we need another "rationality" idea to analyze this type of games.
The first natural class to analyze is that one of the *zero-sum games*: this means that there are two players and in every outcome of the game what one gets is the opposite of what gets the other one. Typically, it is a zero sum game each game where the final situation for the player is either to win, or to tie or else to loose. In this case, as already said, one can conventionally attach 0 utility to the ties, 1 to the victory and -1 in the bad case. Though these are not the most interesting games, at least from the point of view of applications, their study is very useful as reference for the general case.[1]

An $n \times m$ matrix $P = (p_{ij})$, where $p_{ij} \in \mathbb{R}$ for all i, j, represents a two player, finite, zero-sum game in the following sense. Player one chooses a row i, player two a column j, and the entry p_{ij} of the matrix P so determined is the amount the second player pays to the first one.
The first, fundamental, issue is to establish when a pair $(\bar{\imath}, \bar{\jmath})$, i.e. the choice of a row by the first player and of a column by the second one, can be considered as a solution for the game. Let us see an example, from which we try to understand how to proceed.

Example 3.0.1 Consider the game described by the following matrix P:

[1] Somebody made an interesting analogy: the theory of zero sum games represents for the general theory of games what the results on perfect gas in Physics does for the theory of general gas.

$$\begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}.$$

Probably its rational outcome is not immediately seen, but a little analysis will single it out without difficulty. Let us start by looking the game from the point of view of the second player. Clearly, 5 is the maximum amount she agrees to pay because she will pay in any case no more than that by playing the second column (against a possible loss of 8 by playing the two other columns). On the other hand, player one is able to guarantee himself at least 5 (rather than 1 or 0), just playing the second row. As a result, 5 is the outcome of the game.

Generalizing, suppose there exists a value v, row $\bar{\imath}$ and column $\bar{\jmath}$ such that $p_{\bar{\imath}j} \geq v$ for all j and $p_{i\bar{\jmath}} \leq v$ for all i. This means that the first player is able to guarantee himself, by playing row $\bar{\imath}$, *at least* v, while at the same time the second player can guarantee to pay no more than v. This implies in particular (from the first inequality) $p_{\bar{\imath}\bar{\jmath}} \geq v$ and (from the second inequality) $p_{\bar{\imath}\bar{\jmath}} \leq v$: it follows that $p_{\bar{\imath}\bar{\jmath}} = v$ is the rational outcome of the game, with $(\bar{\imath}, \bar{\jmath})$ a pair of optimal strategies for the players.

As suggested by the previous example, in order to find optimal strategies the players must consider the worst outcome for them by selecting each one of their strategies, and then take the strategy providing the best among these worst outcomes. In formulas: if the first player selects the i^{th} row, the worst outcome for him will be $\min_j p_{ij}$. So that the first player will be able to guarantee himself (at least) the quantity $\max_i \min_j p_{ij}$. This is called the *conservative value* of the first player. In the same way, and taking into account a change of sign, the conservative value of the second player will be $\min_j \max_i p_{ij}$. Analyzing the conservative values of the players is then an important issue in this setting.

So, let us consider zero sum games where the strategy sets of the players are not necessarily finite, and let us start by observing the following.

Proposition 3.0.1 *Let X, Y be any sets and let $f : X \times Y \to \mathbb{R}$ be an arbitrary function. Then*

$$\sup_x \inf_y f(x,y) \leq \inf_y \sup_x f(x,y).$$

Proof. Observe that, for all x, y,

$$\inf_y f(x,y) \leq f(x,y) \leq \sup_x f(x,y).$$

Thus

$$\inf_y f(x,y) \leq \sup_x f(x,y).$$

Since the left hand side of the above inequality does not depend on y and the right hand side on x, the thesis easily follows. ■

The inequality provided by the above proposition, a typical inequality in an analytical context, is absolutely natural in view of the interpretation we can give to it within game theory: if f is the payoff function of the first player, and x, y represent the strategies of the first and the second respectively, what the first player can guarantee himself against any possible choice of the second one (the conservative value of the first player, i.e. $\sup_x \inf_y f(x,y)$) cannot be *more* than the maximum amount the second player agrees to pay (no matter the first one does) (the conservative value of the second player, $\inf_y \sup_x f(x,y)$).

The above proposition shows that there is always the same inequality between the conservative values of the players. Of course, sometimes a strict inequality occurs as the following example shows.

Example 3.0.2 The game is described by the following matrix:

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

The matrix is a possible representation of the familiar *scissors, paper, stone* game, with payment 1 to the winner. The conservative value of the first player is -1, of the second is 1. This is the translation, in the mathematical context, of the obvious fact that no matter the players do in this game, they risk to loose: an obvious observation indeed!Our naive intuition that this game cannot have a predictable (rational) outcome is mathematically reflected by the fact that the conservative values of the players are different.

Nevertheless, even games with non coinciding conservative values for the players should not be played totally randomly. For instance, suppose you play the game several times with your sister. If you make her to know that you do not like playing stone (this means that you play stone with probability zero) she will react by playing only scissors, guaranteeing herself at least the draw. Thus in general the players facing these games should choose rows and columns with probabilities suggested by some optimum rule, rather than using always the same strategy. Since the first player has n possible (pure) strategies (the rows of the matrix P), and the second one m pure strategies, the first one will then choose a vector $\alpha = (\alpha_1, \dots, \alpha_n)$ in the n^{th}-simplex,[2] his new strategy space. Similarly, the m^{th}-simplex is the strategy space for the second one. These enlarged strategy spaces are called the spaces of the *mixed strategies* for the players, as we mentioned in the second chapter. The new payment function (what the second one pays to the first) is then the expected value: given strategies α and β,

$$p(\alpha, \beta) = \sum_{i=1,\dots,n, j=1,\dots,m} \alpha_i \beta_j p_{ij} = \langle \alpha, P\beta \rangle = \langle P^t \alpha, \beta \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in Euclidean spaces, and P^t is the transpose of the matrix P.[3] Extending what we did for the finite games, a solution of the game is a pair $(\bar{\alpha}, \bar{\beta})$ of conservative strategies for the players such that there exists v such that

$$p(\bar{\alpha}, \beta) \geq v \quad \forall \beta \quad \wedge \quad p(\alpha, \bar{\beta}) \leq v \quad \forall \alpha.$$

[2] The n^{th}-simplex S_n is the set $S_n = \{\alpha = (\alpha_1, \dots, \alpha_n) : \alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1\}$. α_i represents the probability of playing the row i.

[3] In the notation with brackets, a vector can be indifferently seen as a column or row vector. When multiplying a vector and a matrix we shall instead consider always a vector as a column vector.

It should be noticed that in the case the strategy spaces are not finite, the conservative values are defined in terms of inf and sup, and nor of min and max; moreover, the two values can coincide without having, for one player or both, optimal strategies. In other words, differently from the finite case, in general in order to prove that a game has an equilibrium, one must prove that

- the two values coincide, i.e. $\inf_\beta \sup_\alpha p(\alpha, \beta) = \sup_\alpha \inf_\beta p(\alpha, \beta)$
- there are $\bar{\alpha}$ and $\bar{\beta}$ fulfilling: $\inf_\beta \sup_\alpha p(\alpha, \beta) = \sup_\alpha p(\alpha, \bar{\beta})$, $\sup_\alpha \inf_\beta p(\alpha, \beta) = \inf_\beta p(\bar{\alpha}, \beta)$.

3.1 The minmax theorem by von Neumann

This section is dedicated to one of the master results in Game Theory: it is also, at least in my opinion, one of the most beautiful and elegant results of all mathematics. Though nowadays it can be considered a standard result, which can be explained easily(or almost easily) to master students, it should not be forgot that its proof goes back to less than a century ago, and moreover that some other mathematicians at that time were arguing that a general minimax theorem would not be actually true. Its statement is simple, since it claims that in the extended world including mixed strategies in the games, the matrix games still have equilibria.

Theorem 3.1.1 *A two player, finite, zero sum game as described by a payoff matrix P has equilibrium in mixed strategies.*

The proof of this result is given in the second part of the text, where are developed in detail some prerequisites in Convex Analysis necessary to understand the proof. Moreover, we shall show there how from the theorem of von Neumann the fundamental duality results in Linear Programming can be derived in a smart way, and an algorithm will be developed in order to solve these games.

Here instead we now make some comments and we shall see some consequences of the previous result. First of all, it must be observed that finding optimal strategies of the players by making calculations at hand is not easy, or even impossible: as we shall see, there is no problem in finding solutions when one of the two players has just two strategies, but on the other hand already the 3×3 case can be very annoying to solve. However, there are no problems in translating the problem of finding the optimal strategies in another more familiar one, for which several solvers allow finding (exact) solutions in a short time even for games with many strategies. Let us see how. In order to understand this, it is quite important to make the following observation.

Let us look closely to the payoff function p. Remember that

$$p(\alpha, \beta) = \sum_{i=1,\dots,n, j=1,\dots,m} \alpha_i \beta_j p_{ij} = \langle \alpha, P\beta \rangle = \langle P^t \alpha, \beta \rangle,$$

and I insist on the notation on brackets in order to put in evidence that the function is *linear* with respect one variable, when the other one is fixed. In terms of games, this means that if I am the first player, and I know the strategy used by the second one, my best choice to it can be always chosen among the *pure strategies*. In particular my best choice can be a certain number of pure strategies, and any mixture among them. This corresponds mathematically to the fact that when minimizing or maximizing a linear function on a simplex a solution can be always found on the vertices. Think at the simplest case, when one has to minimize or maximize a linear (or affine) function on a segment: of course the minimum and the maximum are at the extreme points of the segment. It can happen also that the function is constant in the segment, and thus the minimum/maximum can be also in the other points of the segment. This can be generalized to any dimension: of course we loose the geometrical vision when we are in more dimensions, but the idea is exactly the same.

Having this in mind, we know that the first player must choose a probability distribution $\alpha = (\alpha_1, \dots, \alpha_n)$ in S_n in such a way that:

$$\begin{array}{l} \alpha_1 p_{11} + \cdots + \alpha_n p_{n1} \geq v \\ \dots \\ \alpha_1 p_{1j} + \cdots + \alpha_n p_{nj} \geq v \\ \dots \\ \alpha_1 p_{1m} + \cdots + \alpha_n p_{nm} \geq v, \end{array} \tag{3.1}$$

where v must be as large as possible. This is because the amount

$$\alpha_1 p_{1j} + \cdots + \alpha_n p_{nj}$$

is what he will get if player two chooses the column j. Thus the constraint set we impose has the meaning that he will gain at least v, no matter which column will be played by the opponent and thus, no matter which *probability distribution* she will choose on the columns. And obviously, player one is interested in maximizing v. The second player instead has to find $S_m \ni \beta = (\beta_1, \dots, \beta_m)$ such that:

$$\beta_1 p_{i1} + \cdots + \beta_m p_{im} \leq u, \quad 1 \leq i \leq n,$$

where u must be as small as possible.

Let us now see how it is possible to use a trick in order to write the above problems in a more familiar form. First of all, we suppose, without loss of generality, that all coefficients of the matrix are positive.[4] Thus the value of the game is surely strictly positive and so it is enough to make a change of variable, by setting $x_i = \frac{\alpha_i}{v}$. Condition $\sum_{i=1}^m \alpha_i = 1$ becomes $\sum_{i=1}^m x_i = \frac{1}{v}$. Then maximizing v is equivalent to minimizing $\sum_{i=1}^m x_i$.

Thus, denoting by 1_j the vector in $\mathbb{R}^j$ whose all coordinates are 1, we can write the first player problem in the following way:

$$\begin{cases} \min\langle 1_n, x\rangle \text{ such that} \\ x \geq 0, P^T x \geq 1_m \end{cases}. \tag{3.2}$$

In the same way, we see that the second player faces the following problem:

[4] If this is not so, we can add a large constant a to all coefficients. As it is easily seen, this makes the payoff function to change from $p(\alpha, \beta)$ to $p(\alpha, \beta) + a$. Of course this does not change the optimal strategies for the players, only the value of the game is augmented by a. You can explain the players that adding this is a good trick, and that they can accept it, with the convention that the first player will offer the quantity a to the second player in order to convince her to change the matrix.

$$\begin{cases} \max\langle 1_m, y\rangle \text{ such that} \\ y \geq 0, Py \leq 1_n \end{cases}. \tag{3.3}$$

These are two linear problems in duality, so that the well known machinery to solve them can be used also to find optimal strategies for the players in zero sum games. Furthermore, another algorithm can be used, efficient and simple at least for games of small dimensions.

We conclude this section with some remark on a special class of zero sum games.

Definition 3.1.1 *A square matrix $n \times n$ $P = (p_{ij})$ is said to be* antisymmetric *provided $p_{ij} = -p_{ji}$ for all $i, j = 1, \dots, n$.*
A (finite) zero sum game is said to be fair *if the associated matrix is antisymmetric.*

The meaning of the above definition is clear: the two players have same options and same utilities. For, what the first faces when playing the i-th row is the same of what the second one faces by playing the i-th column (of course a change of sign is present in the entries because of the convention on the payments). It is very intuitive to expect that the value of this category of games should be zero, and that the two players should have the same optimal strategies. This is also true and easy to prove.

Proposition 3.1.1 *If the square matrix $n \times n$ $P = (p_{ij})$ is said antisymmetric the value is* 0 *and p is an optimal strategy for player 1, if and only if it is so for player 2 as well.*

Proof. Since

$$\alpha^t P\alpha = (\alpha^t P\alpha)^t = \alpha^t P^t\alpha = -\alpha^t P\alpha,$$

it follows that $f(\alpha, \alpha) = 0$ for all α and thus $\inf_\beta f(\alpha, \beta) \leq 0$. This implies that the conservative value of the first player is non positive. Analogously the conservative value of the second player is non negative. Since they agree by the von Neumann theorem, the value is zero. Suppose now α optimal for the first player. Then it must be $\alpha^t P\beta \leq 0$ for all β. Since $\alpha^t P\alpha = 0$, then α optimal for the second player. ■

3.2 The Indifference Principle

In this section we want to see the so called *indifference principle*, a useful and illuminating idea, at least from a theoretical point of view. Moreover, sometimes it allows finding optimal strategies in a fast way. To understand this, let us take a closer look at the inequalities of the system (3.1) above, taking for granted that the value of the game is v. Suppose moreover $\bar{\alpha}$ is an optimal strategy for the first player. The question we want to investigate is at what extent we can investigate the presence of equalities and strict inequalities in the system

$$\begin{array}{l} \bar{\alpha}_1 p_{11} + \dots + \bar{\alpha}_n p_{n1} \geq v \\ \dots \\ \bar{\alpha}_1 p_{1j} + \dots + \bar{\alpha}_n p_{nj} \geq v \\ \dots \\ \bar{\alpha}_1 p_{1m} + \dots + \bar{\alpha}_n p_{nm} \geq v \end{array} \tag{3.4}$$

The first remark is the following: *not all inequalities can be strict inequalities.* This is immediate to understand. For, suppose there is an optimal strategy for the second player using the j-th column with positive probability. Then it must be

$$\bar{\alpha}_1 p_{1j} + \cdots + \bar{\alpha}_n p_{nj} = v,$$

otherwise the first player will get *more* than v in the game (at least v against all other pure strategies and more than v against the j-th strategy implies an expected value for the first player of more than v). Thus form this we can immediately conclude that there is equality in

$$\bar{\alpha}_1 p_{1j} + \cdots + \bar{\alpha}_n p_{nj} \geq v,$$

at least for each index (column) j such that there exists an optimal strategy for the second player using column j with positive probability. It is also true the opposite: if the second player in no optimal strategy assigns positive probability to a column j, then there is an optimal strategy for the first player guaranteeing him *more* than v against the j-th column. However this is less easy to prove and we do not see its proof here (however see the chapter on zero sum games in the second part).
In particular, even if we do not know the actual value of the game, we can conclude that *every strategy used with positive probability by the second player at an equilibrium will provide the same value to the first player.* This explains the name of indifference principle. For, the first player must be indifferent with respect to all strategies effectively used by the second one at an equilibrium.
However this principle, though very important, cannot be extensively used in practice, for a very simple reason: in general the strategies used by a player to an equilibrium are not known a priori. So that, we cannot in general know where in the system (3.4) we do actually have equality. With an exception: in the 2×2 games, when chasing for mixed equilibrium strategies, the indifference principle is the fastest way to do the job. Let us see an example:

Example 3.2.1 let the game be described by the following matrix P:

$$P = \begin{pmatrix} 3 & -2 \\ -3 & 4 \end{pmatrix}.$$

If is easy to see that there are no equilibria in pure strategies. Suppose now I plays $(p, 1-p)$, where $0 < p < 1$. The second one will get $3p - 3(1-p)$ from the first column, and $-2p + 4(1-p)$ from the second one. By imposing equality we get $p = \frac{7}{12}$. Doing the same to find the optimal strategy $(q, 1-q)$ of the second player, we get $q = \frac{1}{2}$. And the value of the game is easily calculated as well. For instance from the first row,

$$\frac{7}{12}\frac{1}{2}3 + \frac{5}{12}\frac{1}{2}(-2) = \frac{11}{24}.$$

3.3 Exercises

Exercise 3.3.1 Solve the following zero sum game

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 7 & 1 \\ 1 & 0 & 5 \end{pmatrix}.$$

Solution This is the typical example of game which can be solved by elimination of dominated strategies.
We notice that the third column is dominated by the first one, as 4 is bigger than 2, 1 bigger than 0 and 5 is bigger than 1. The second player does not play a strategy which is always worst for her (as she always has to pay more) then we can eliminate the last column and we obtain

$$\begin{pmatrix} 2 & 3 \\ 0 & 7 \\ 1 & 0 \end{pmatrix}.$$

Then we observe that the last row is dominated by the first row and we have

$$\begin{pmatrix} 2 & 3 \\ 0 & 7 \end{pmatrix}.$$

now the second column is dominated

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

and finally the second row is dominated too. The result is 2: the first player plays the first row, the second player the first column and the result is that the second player pays 2 to first one.

Exercise 3.3.2 Find the saddle points of the following matrix:

$$A = \begin{pmatrix} 9 & 6 & 11 & 7 & 6 & 8 \\ 3 & 0 & 6 & 11 & 3 & 2 \\ 8 & 6 & 12 & 7 & 6 & 7 \\ 0 & 1 & 10 & 3 & 2 & 6 \end{pmatrix}.$$

Solution Player I can obtain at least 6 playing the first row, 0 playing the second , 6 from the third and 0 from the fourth. The maximum is 6, then the conservative value of the first player is $v_I = 6$ playing the first or the third row. Similarly the conservative value of the second player is $v_{II} = 6$ playing the second or the fifth column.
$v = 6$ is the value of the game and p_{12}, p_{15}, p_{32} and p_{35} are equilibria. Observe p_{23} and p_{46} are *not* equilibria.

Exercise 3.3.3 Two friends play the following game: both players have a sheet of paper. One paper contains the number 7 in red on one side and 3 in black on the other side, the other one 6 in red and 4 in black. They must show at the same time one side of the paper: if the two colors agree player having 7 and 3 wins the number written in other player's paper. The opposite if the colors disagree. What sheet of paper would you like to have?

Solution The game is described by the following matrix

$$P = \begin{pmatrix} 6 & -7 \\ -3 & 4 \end{pmatrix}$$

in which the first row and column correspond to the strategy of showing the red side of the paper, the second ones the black side.

The conservative values for the players are -3 and 4, then there is no equilibrium in pure strategies. If I plays $(p, 1-p)$, from the indifference principle we get $6p-3+3p = -7p+4-4p$ implying $p = \frac{7}{20}$. In the same way we obtain that if player II plays $(q, 1-q)$ we get $6q-7+7q = -3q = 4-4q$ and her strategy is $q = \frac{11}{20}$. The value of the game is $v = \frac{3}{20}$ and then it is more convenient to be player I.

Exercise 3.3.4 Given the zero sum game described by the following matrix:

$$P = \begin{pmatrix} 6 & 0 & 5 & 3 \\ 1 & 5 & 4 & 4 \end{pmatrix}$$

find a pair of optimal strategies for the players and the value of the game.

Solution A convex combination of the two first columns (weakly) dominates the other two columns (e.g. $\left(\frac{5}{12}, \frac{7}{12}\right)$). From the first two, if player I plays $(p, 1-p)$, from the indifference principle we get $6p + 1 - p = 5 - 5p$ implying $p = \frac{2}{5}$. Similarly the strategy for II is $q = \frac{1}{2}$ and the value of the game is $v = 3$.

Exercise 3.3.5 Given the zero sum game described by the following matrix:

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & b \end{pmatrix},$$

1. find all a, b such that there are equilibria in pure strategies
2. find the value and all equilibria in mixed strategies for $b < 0$ and $a > 0$.

Solution

1. Playing the first row, player I can obtain at least 0 if $a \geq 0$, a otherwise. She can obtain at least 0 from the second row and from the third row 0 if $b \geq 0$ and b otherwise. The conservative value of the first player is $v_I = 0$ for each a and b. There are equilibria in pure strategies if the conservative value of the second player is also 0, i.e. if $a, b < 0$
2. Suppose $b < 0$ and $a > 0$. The third row is dominated and the game reduces to: $\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \end{pmatrix}$. If $0 < a \leq 1$, last column dominates the first one and the matrix of the game is reduced to $\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$. If I plays $(p, 1-p, 0)$ from the indifference principle we get $1 - p = ap$ then $p = \frac{1}{a+1}$, if II plays $(0, q, 1-q)$ we get $a - aq = q$ then $q = \frac{a}{a+1}$ and the value is $v = \frac{a}{a+1}$.
   If $a > 1$, the last column is dominated and the matrix reduces to
   $$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
   the value is $v = \frac{1}{2}$ and the two players play $\left(\left(\frac{1}{2}, \frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}, 0\right)\right)$.
   If $a = 1$, the two players play a combination of $\left(\lambda, \frac{1}{2}, \frac{1}{2} - \lambda\right)$ with $0 \leq \lambda \leq \frac{1}{2}$.

Exercise 3.3.6 Given the zero sum game described by the following matrix:

$$\begin{pmatrix} 7 & 4 & 3 \\ 1 & 4 & 5 \end{pmatrix},$$

1. find the conservative values of the players and say if there are equilibria in pure strategies
2. find the value and all equilibria in mixed strategies.

Solution

1. The conservative values of the players are $v_I = 3$ and $v_{II} = 4$. As $v_I \neq v_{II}$, there are no equilibria in pure strategies
2. Representing the strategies of the second player, we can observe that the value of the game is 4. No column is dominated and an optimal strategy for the second player is then given by $(p, q, 1-p-q)$ s.t. $7p + 4q + 3(1-p-q) = 4$, i.e. s.t. $4p + q = 1$. An optimal strategy for the first player is given by playing $(p, 1-p)$ s.t. $7p + 1 - p = 4$, then $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Exercise 3.3.7 Given the zero sum game described by the following matrix:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

with $a, b, c > 0$, find the equilibrium of the game.

Solution As obviously no row/column may be used with probability zero, we apply the indifference principle. A strategy (x_1, x_2, x_3) for the first player has to satisfy the following conditions

$$\begin{cases} ax_1 = v \\ bx_2 = v \\ cx_3 = v \\ x_1 + x_2 + x_3 = 1. \end{cases} .$$

and we get $v = \frac{abc}{bc+ac+ab}$ and $(x_1, x_2, x_3) = \left(\frac{bc}{bc+ac+ab}, \frac{ac}{bc+ac+ab}, \frac{ab}{bc+ac+ab}\right)$. Similarly, we obtain $(y_1, y_2, y_3) = \left(\frac{bc}{bc+ac+ab}, \frac{ac}{bc+ac+ab}, \frac{ab}{bc+ac+ab}\right)$ for the second player.

Exercise 3.3.8 The duel Two players fight a duel with one shot pistols. Their strategy set is the moment when shooting. They are at distance 2 and they advance at constant speed 1. The probability for both is the same and it is given by the distance from the starting point (so that the strategy space is $[0, 1]$ for both). Discuss the value of the game, in the case they either can or cannot understand if the opponent has fired (and they are not hit, of course).

Solution In the first care, if $p < q$ player I is the first to shoot, she has probability p to kill player II, otherwise, with probability $1-p$, she will die, as player II waits to be sure to kill her before shooting, then

$$f(p, q) = p \cdot 1 + (1-p) \cdot (-1) = p - 1 + p = 2p - 1.$$

If $p > q$ the situation is symmetric and

$$f(p, q) = q \cdot (-1) + (1-q) \cdot 1 = -q + 1 - q = 1 - 2q.$$

Then

$$f(p,q) = \begin{cases} 2p-1 & p<q \\ 0 & p=q \\ 1-2q & p>q \end{cases}$$

As $p > q \Rightarrow 1-2q > 1-2p$

$$\inf_{0\leq q\leq 1} f(p,q) = \min\{2p-1, 0, 1-2p\} = -|1-2p|$$

and

$$\sup_{0\leq p\leq 1} -|1-2p| = 0,$$

from which we get $p = 1/2$.
Similarly

$$\sup_{0\leq p\leq 1} f(p,q) = \max\{2q-1, 0, 1-2q\} = |1-2q|$$

and

$$\inf_{0\leq q\leq 1} |1-2q| = 0,$$

then the value of the game is 0 and $q = 1/2$.

In the second case, if $p < q$, player I is the first to shoot, she has probability p to kill player II, otherwise, player II kills her with probability q, then

$$f(p,q) = p\cdot 1 + (1-p)\cdot q\cdot(-1) = p - q + pq$$

If $p > q$ the situation is symmetric and

$$f(p,q) = q\cdot(-1) + (1-q)\cdot p\cdot 1 = p - q - pq$$

$$f(p,q) = \begin{cases} p-q+pq & p<q \\ 0 & p=q \\ p-q-pq & p>q \end{cases}$$

As $p < q \Rightarrow p - q + pq > 2p - 1$ and $p > q \Rightarrow p - q - pq > -p^2$

$$\inf_{0\leq q\leq 1} f(p,q) = \min\{2p-1, 0, -p^2\}$$

The value of the first player is given by the intersection of $2p-1$ and $-p^2$ and it is equal to $2\sqrt{2}-3$. As the game is symmetric, if it has a value it has to be 0, this implies that the game does *not* have value.

Exercise 3.3.9 Given the zero sum game:

$$P = \begin{pmatrix} 5 & 2 & 1 \\ 3 & 6 & 7 \end{pmatrix},$$

1. find the conservative values of the players and say if there are equilibria in pure strategies
2. find the value and all equilibria in mixed strategies.

Exercise 3.3.10 Given the zero sum game:

$$P = \begin{pmatrix} 5 & 3 & 3 \\ 3 & 3 & 5 \end{pmatrix},$$

1. find the conservative values of the players and say if there are equilibria in pure strategies
2. find the value and all equilibria in mixed strategies.

Exercise 3.3.11 Consider the following game: Emanuele and Alberto must show each other one or two fingers and tell a number, at the same time, trying to guess the number of the fingers shown. If both are right or wrong, they get zero. If one is wrong and the other one is right, who is right gets the number he said. Tell what they play.

4

The Non Cooperative Strategic Approach

4.1 Games in strategic form

It is quite clear that there are many situations where two people can take decisions advantaging (or damaging) both, as famous examples like the prisoner dilemma or the battle of sexes show; in this case the theory of zero sum games is useless. Why is it so? Let us look at the following example:

Example 4.1.1 Consider the bimatrix:

$$\begin{pmatrix} (10,10) & (-1,1) \\ (0,0) & (1,-2) \end{pmatrix}.$$

The security levels, given by the maxmin strategies, for both players are easily calculated. If they use them, the outcome of the game would be $(0,0)$. This is a non sense, the outcome $(10,10)$ looks much more reasonable.

Observe that in the above example, the natural rational outcome cannot be singled out by means of the procedure of eliminating dominated strategies. With a little thinking, one can easily realize that the above cannot be the strategic form of a game of a perfect information, and thus the backward induction cannot be applied. Thus, once again, we need to appeal to something different as a rationality paradigm in these more general situations.[1]
After the contributions of von Neumann to the zero sum theory, the next step to the mathematical understanding of the games was the famous book by von Neumann-Morgenstern, *The theory of games and economic behaviour*, whose publication was later taken as the official date of the birth of game theory. There, in order to handle the situations which cannot be modelled as zero sum games, a cooperative approach

[1] Remember: the first rationality rule was elimination of dominated strategies. Then for games with perfect recall we proposed backward induction, that however cannot be applied to the case of games with contemporary moves. For handling this case we considered at first the special class of the zero sum games, and we argued that saddle points (which are related to the conservative values of the players) are the good idea for rationally solving the game. Now we want to make a further step to include also non strictly competitive games.

has been developed. It was an attempt to study the mechanisms of interaction between agents having different, but not necessarily opposite, interests. This theory, tough interesting, was not considered fully satisfactory, at least at the beginning, for (economical) applications, especially for the difficulty to handle (and sometimes to understand) the "outcome" of the game there considered. At the beginning of the fifties of the last century, J. F. Nash proposed a different model, and a new idea of equilibrium, which was almost immediately considered well suited for economical applications. Here it is, written for two players for simplicity of notation.

Definition 4.1.1 *A* two player noncooperative game in strategic (or normal) form *is a quadruplet* $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$. *A* Nash equilibrium *for the game is a pair* $(\bar{x}, \bar{y}) \in X \times Y$ *such that:*

- $f(\bar{x}, \bar{y}) \geq f(x, \bar{y})$ for all $x \in X$
- $g(\bar{x}, \bar{y}) \geq g(\bar{x}, y)$ for all $y \in Y$.

We have already introduced this definition of game. Remember, X and Y are the strategy spaces of player one and two respectively. Every pair (x, y), when implemented, gives raise to a result which provides utility $f(x, y)$ to the first player, and $g(x, y)$ to the second one. An equilibrium point is a pair with the following feature: suppose somebody proposes the pair $(\bar{x}, \bar{y})$ to the players. Can we expect that they will object to it? The answer is negative, anytime $(\bar{x}, \bar{y})$ is a Nash equilibrium, because each one, *taking for granted that the other one will play what he was suggested to play,* has no incentive to deviate from the proposed strategy. On the other hand, it is rational to take for granted that the other player will not deviate, for the same reason. A simple idea, worth a Nobel Prize.

It is very important now to understand if the solution concepts we introduced before fit in this idea of Nash equilibrium. First of all, let us consider the criterion of eliminating dominated strategies. This says that we cannot use a strategy x for the first player if there exists another strategy z such that $f(x, y) > f(z, y)$ for all $y \in Y$. Clearly, in this way we cannot eliminate Nash equilibria.[2] And in the case there is (strictly) dominant strategy x for player one, this means that

$$f(x, y) > f(z, y) \qquad \forall y, \forall z \neq x,$$

and it is apparent how this condition is much more stringent than the condition relative to the first player in the definition of Nash equilibrium: there it is required only

$$f(x, \bar{y}) \geq f(z, \bar{y}) \forall z \in X,$$

i.e. the inequality is required *only* for $\bar{y}$ and not for all y: a crucial difference making the dominant strategy a very special case of a Nash equilibrium. As a conclusion, if a player has a strictly dominant strategy, there is a Nash equilibrium, and all equilibria have the dominant strategy relative to that player, paired to all strategies maximizing the utility of the other player, once fixed the dominant strategy in his utility function. A bit more complicated is the case of backward induction, at least to state formally, even if it is very intuitive that the procedure will definitely single out Nash equilibria: the procedure itself is based on systematic optimal behavior of the player (when

[2] A different situation can arise if one eliminates also weakly dominated strategies.

subsequent choices are determined). Let us try to explain in words the idea. The proof can be made by induction on the length of the game. The games of length 1 are trivial, since only one player has available moves: he will choose the branch leading him the best outcome, and this is clearly a Nash equilibrium (the utility functions of the players depend *only* on the variable relative to one player). Now suppose to have proved that for all games of length $n-1$ the backward induction procedure provides a Nash equilibrium. How do we see that it induces a Nash equilibrium also in games of length n? Well, we distinguish two cases. If at the root of the game it is player one to take a decision, its backward induction strategy tells her to play the branch leading to the game(s), of length at most $n-1$, giving her the best utility: in this case it is clear that once again we have a Nash equilibrium, since she is taking the best decision for her at the root, and by inductive assumption she is selecting a Nash equilibrium in the subgame chosen at the first level (actually at every subgame, not only at the selected one). For the other players the same argument applies: the collection of their strategies in the subgames is an optimal decision for them by inductive assumption. In the case the nature is making a decision at the root, once again the the collection of the strategies in the subgames is an optimal decision for each player by inductive assumption. Thus, backward induction actually selects a Nash equilibrium. Even more, the above argument and intuition suggest that other Nash equilibria can arise, since Nash equilibrium requires optimality along the path effectively followed by the players, while backward induction requires optimality at any subgame: we shall see an example of this later. Finally, let us consider the zero sum case. The next theorem, though very simple, tells us interesting things.

Theorem 4.1.1 *Let X, Y be (nonempty) sets and $f : X \to \mathbb{R}$ a function. Then the following are equivalent:*

1. *The pair $(\bar{x}, \bar{y})$ fulfills*

$$f(x, \bar{y}) \leq f(\bar{x}, \bar{y}) \leq f(\bar{x}, y) \quad \forall x \in X,\ \forall y \in Y$$

2. *The following conditions are satisfied:*
(i) $\inf_y \sup_x f(x, y) = \sup_x \inf_y f(x, y)$*;*
(ii) $\inf_y f(\bar{x}, y) = \sup_x \inf_y f(x, y)$*;*
(iii) $\sup_x f(x, \bar{y}) = \inf_y \sup_x f(x, y)$.

Proof. Let us begin by seeing that 1. implies 2. From 1. we get:

$$\inf_y \sup_x f(x, y) \leq \sup_x f(x, \bar{y}) = f(\bar{x}, \bar{y}) = \inf_y f(\bar{x}, y) \leq \sup_x \inf_y f(x, y).$$

From Proposition 3.0.1 we can conclude that in the line above all inequalities are equalities, and thus 2. holds. Conversely, suppose 2. holds. Then

$$\inf_y \sup_x f(x, y) = (iii) = \sup_x f(x, \bar{y}) \geq f(\bar{x}, \bar{y}) \geq \inf_y f(\bar{x}, y) = (ii) = \sup_x \inf_y f(x, y).$$

So that, because of (i), we have all equalities and the proof is complete. ■

The above theorem is quite interesting from the point of view of its consequences: first of all it puts in evidence that a *saddle* point $(\bar{x}, \bar{y})$ as in condition 1. is an *equilibrium* of the game, since the choice of $(\bar{x}, \bar{y})$ guarantees the players to reach their conservative values (that agree according to condition 2.): but then, remembering that

$g = -f$, the outcome proposed by von Neumann for these games is actually a Nash equilibrium, as condition 1. shows. There is much more. Condition 2. shows that the players must solve two *independent* problems in order to find their optimal strategies. So that, they do not need to know what the opponent will do. Condition 2. tells us one more interesting thing: if (x, y) and (z, w) are two saddle points, then also (x, w) and (z, y) are saddle points and f takes the same value at the saddle points: the so called *rectangular* property of the saddle points. This means that the two players *do not need* to coordinate their strategies. This was already made apparent by the fact that finding optimal strategies for the players is equivalent to solve linear programming problem, but it is very useful to remember it here, in that this is no longer true in general games: looking at the battle of sexes we see that the players must coordinate their choices, since both their strategies can produce an equilibrium, but *only* if in agreement with a strategy of the other player.[3] To conclude, observe that a player can take a unilateral decision bringing him to a Nash equilibrium, outside the zero sum case, when she has a strictly dominant strategy, but this is an exceptional case.

Let us come back to the general setting and try to understand how to characterize Nash equilibria. What does the rational player one, once he knows (or believes) that the second player plays a given strategy y? Clearly, he maximizes his utility function $x \mapsto f(x, y)$, i.e. he wants to choose a strategy x belonging to $\text{Max}\,\{f(\cdot, y)\} = \{x \in X : f(x, y) \geq f(u, y)\ \forall u \in X\}$. Denote by BR_1 the following multifunction:

$$BR_1 : Y \to X : \quad BR_1(y) = \text{Max}\,\{f(\cdot, y)\}$$

(BR stands for "best reaction"). Do the same for the second player and finally define:

$$BR : X \times Y \to X \times Y : \quad BR(x, y) = (BR_1(y), BR_2(x)).$$

Then it is clear that a Nash equilibrium for a game is nothing else than a fixed point for BR: $(\bar{x}, \bar{y})$ is a Nash equilibrium for the game if and only if

$$(\bar{x}, \bar{y}) \in BR(\bar{x}, \bar{y}).$$

Thus a fixed point theorem will provide an existence theorem for a Nash equilibrium.

Let us recall here the Kakutani fixed point theorem:

Let Z be a compact convex subset of an Euclidean space, let $F : Z \to Z$ be a nonempty closed convex valued multifunction[4] with closed graph. Then F has a fixed point: there is $\bar{z} \in Z$ such that $\bar{z} \in F(\bar{z})$.

Let us now state an existence theorem for Nash equilibria of a game:

Theorem 4.1.2 *Given the game* $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$*, suppose* X*,* Y *convex closed bounded subsets of some finite dimensional vector space, suppose* f, g *continuous and, moreover:*

- $x \mapsto f(x, y)$ *is quasi concave for all* $y \in Y$*;*

---

[3] Differently and more simply said, for me both going to see a movie or to watch a game can be part of a Nash equilibrium, but only at the condition that my girlfriend goes to the same place.

[4] This means that $F(z)$ is a nonempty closed convex set for all z.

- *$y \mapsto g(x, y)$ is quasi concave for all $x \in X$.*

Then the game has an equilibrium.

Proof. Remember that quasi concavity of a function h is by definition convexity of the a level sets h_a of h: $h_a = \{z : h(z) \geq a\}$. Starting from this, it is very easy to see that BR is nonempty (due to the compactness assumption) closed convex valued (here quasi concavity of the payoff functions and convexity of the domains play a role). Checking that BR has closed graph is equally easy: suppose $(u_n, v_n) \in M(x_n, y_n)$ for all n and $(u_n, v_n) \to (u, v)$, $(x_n, y_n) \to (x, y)$. We must show that $(u, v) \in M(x, y)$. We have that

$$f(u_n, y_n) \geq f(z, y_n), \qquad g(x_n, v_n) \geq g(x_n, t),$$

for all $z \in X$, $t \in Y$. Now pass to the limit using continuity of f and g to get the result. ∎

In the following example we make explicit the fact that backward induction does not find all Nash equilibria of a game of perfect information, written in strategic form.

Example 4.1.2 (The ice cream game) Mom, the player number one, is with her son, the player number two, at the shopping center. The son wants an ice cream. Mom does not like too much to buy it, since she thinks that too many ice creams will increase the bill of the dentist. On the other hand, the son knows that he could cry to be more convincing, even if he does not like very much to do it. Thus the first player has two moves, to buy or not to buy the ice cream. The second one can decide, after the decision of his mom, whether to cry or not. Let us quantify their utility functions. If mom buys the ice cream, her utility is 0, while the utility of the son is 20. If she announces that she does not buy the ice cream, the son can cry, and the utilities are, in this case, -10 for mom and -1 for the son, while if he does not cry, the utilities are 1 for both. It is very easy to see that the outcome $(1, 1)$ is the only one the backward induction selects, while $(0, 20)$ (strategies: mom buys, the son announces to cry) is another outcome supported by a Nash equilibrium in the normal form of the game.

It is very clear that a bimatrix game need not to have a Nash equilibrium. To be convinced of this, without even writing a bimatrix, is enough to think of a *matrix* game without saddle points: it is as well an example of a *bimatrix* game without Nash equilibria.
However it is also clear that a bimatrix game will have Nash equilibria, in mixed strategies (actually Nash stated his theorem exactly in this setting). In this case it can be profitable to use, sometimes, the *indifference principle*, introduced for zero sum games, but whose validity extends to this more general setting.

Consider, for simplicity, a bimatrix game with two rows and two columns, and let us look for *completely* mixed strategies for the two players. In other words, let us suppose the two rows (columns) will be used with positive probability. It is clear that, once fixed the strategy of one player, say player one, the best reaction multifunction of the other player, in our case player two, will contain also a pure strategy, i.e. a column. Thus, when looking for mixed equilibria, the players must be indifferent to the strategy announced by the opponents. Let us see it with the help of an example.

Example 4.1.3 Let us consider the game described by the following bimatrix:

$$\begin{pmatrix} (1,0) & (0,3) \\ (0,2) & (1,0) \end{pmatrix}.$$

There are not pure Nash equilibria. Denote by $\bar{p}$ and $\bar{q}$ respectively, the probability used by the players at the equilibrium, when using the first row (column). When the second player uses $(\bar{q}, 1-\bar{q})$, the first one must be indifferent between the first and second row. Thus:

$$\bar{q}1 + (1-\bar{q})0 = \bar{q}0 + (1-\bar{q})1,$$

providing $\bar{q} = \frac{1}{2}$. Analogously, we get $\bar{p} = \frac{2}{5}$. Thus we get:
Nash equilibria

$$[(\frac{2}{5}, \frac{3}{5}), (\frac{1}{2}, \frac{1}{2})],$$

with payments

$$(\frac{1}{2}, \frac{6}{5}).$$

You should try to verify the result by calculating the best reaction multifunctions.[5]

The above example is very useful for showing another interesting fact. Even if the conservative values of the players in non zero sum games have less importance than in the zero sum case, it is interesting for the players to compute it, in order to know how much they are able to get from the game, no matter what the other player does. We leave as an exercise to verify that the security level for the players are

$$[(\frac{1}{2}, \frac{1}{2}), (\frac{3}{5}, \frac{2}{5})],$$

with payments

$$(\frac{1}{2}, \frac{6}{5}).$$

Make a comparison with the result concerning the Nash equilibria. First of all, the payments are the same for both! Secondly, the relative strategies (Nash versus conservative) are different for both! Once again, a paradoxical situation. What should you do? Playing the Nash equilibrium means that you believe that your opponent is rational and cannot make mistakes, which is not always true... on the other hand, playing the conservative strategy, in order to be sure to achieve at least what you can get from the Nash equilibrium, is not stable, since the conservative strategies are not best reply for the players!

Finally, let us observe that in any game, at any Nash equilibrium, the players get at least what they would get by playing their conservative strategies. For, if $(\bar{x}, \bar{y})$ is a Nash equilibrium of the game, for the first player we have: $\forall x$,

$$f(\bar{x}, \bar{y}) \geq f(x, \bar{y}) \geq \min_y f(x, y),$$

whence

$$f(\bar{x}, \bar{y}) \geq \max_x \min_y f(x, y).$$

The same argument applies to the second player.

Games where the players at any equilibrium can get no more than the conservative values are defined in the literature *non profitable games.*

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[5] It should be remembered that by using the indifference principle some Nash equilibria could be missing. An example of this is given in Exercise 4.5.21.

To conclude this part, let us say that for general games in normal form the "rationality" principle we need in order to build up the theory can in this case be stated as "an outcome of the game is rational if and only if it is a Nash equilibrium". This is generally accepted even if it is clear that now rationality is much less evident than for instance in the principle of elimination of dominated strategies, which however does not usually allow to select some outcome for the game. Moreover, as we have seen in an example, especially when games in extensive form are considered, not all Nash equilibria seem to be really satisfactory outcomes of the game. For this reason, other more refined concepts of equilibria are present in the literature, like trembling hand equilibria, perfect equilibria, sequential equilibria and so on. However it is quite clear that the idea of best reaction is at the core of the idea of equilibrium for non cooperative games in strategic form, as we shall also see soon when talking about correlated equilibria, a quite interesting extension of the idea of Nash equilibrium.

4.2 Some models of duopoly

In this section we analyze some different models of the situation when two firms are present in the market. This is called the model of the *duopoly*. The difference with respect to the case when many firms are present is that in the former case the firms are *price makers*, in the second *price takers*. Though the models presented are very naif, they show some facts which are very reasonable and interesting from an economical perspective.

4.2.1 The Cournot model

In this model, the firms choose quantities to produce. We assume that firm 1 produces the quantity q_1, firm 2 produces quantity q_2, the unitary cost is the same, $c > 0$, for both, and that a quantity $a > 0$ of the good saturates the market.[6] Thus, we can assume that the price $p(q_1, q_2)$ is given by $p = \max\{a - (q_1 + q_2), 0\}$. Thus the interval $[0, a]$ can be considered the strategy space for both players. Now the payoffs:

$$u_1(q_1, q_2) = q_1 p(q_1, q_2) - cq_1 = q_1(a - (q_1 + q_2)) - cq_1,$$
$$u_2(q_1, q_2) = q_2 p(q_1, q_2) - cq_2 = q_2(a - (q_1 + q_2)) - cq_2.$$

We make the (obvious) assumption that $a > c$, otherwise there is no market. First of all, let us see what happens if one firm does not produce, i.e. the monopolistic case. This means assuming $q_2 = 0$ and firm 1 maximizes $u(q_1) = q_1(a - q_1) - cq_1$. This provides:

$$q_M = \frac{a-c}{2}, \quad p_M = \frac{a+c}{2} \quad u_M(q_M) = \frac{(a-c)^2}{4}.$$

Now, let us find the best reaction multifunctions of the players, in the case of duopoly. Being the utility functions strictly concave and non positive at the endpoints

[6] This is translated in the model by setting the price 0 when the quantity a is put on the market. We shall also assume that the price is linear in the quantity produced.

of the domain, we can get it by making the first derivative vanish. We get, for the two players, respectively:

$$a - 2q_1 - q_2 - c = 0, \qquad a - 2q_2 - q_1 - c = 0,$$

and solving the system we get:

$$q_i = \frac{a-c}{3}, \quad p = \frac{a+2c}{3} \quad u_i(q_i) = \frac{(a-c)^2}{9}.$$

Making a comparison with the case of a monopoly, we see that:

- the price is lower in the duopoly case;
- the total quantity of product in the market is superior in the duopoly case;
- the total payoff of the two firms is less than the payoff of the monopolist.

In particular, the two firm could consider the strategy of equally sharing the payoff of the monopolist, but this is not an equilibrium! The result shows a very reasonable fact, the consumers are better off if there is no monopoly.

A further interesting remark is that the Nash equilibrium can be seen as the result of eliminating dominated strategies. Let us see why: first of all, observe that the players are symmetric. So what applies to one, applies to the other one as well. As a first step, we see that the first player will never play more than $\frac{a-c}{2}$, since if she does so derivative $(a-c-q_2) - 2q_1$ of her payoff function is negative for all q_2. Thus the choice of $\frac{a-c}{2}$ is better off, no matter is the choice of q_2 of a bigger quantity. This applies to the second player as well, and thus $q_2 \leq \frac{a-c}{2}$. Since the derivative of the payoff function of the second player is positive in the interval $[0, \frac{a-c}{4}]$ for all $q_2 \leq \frac{a-c}{2}$, we see that all strategies in the interval $[0, \frac{a-c}{4})$ are dominated by $\frac{a-c}{4}$). Thus the strategy set for the player is now reduced to the interval $[\frac{a-c}{4}, \frac{a-c}{2}]$: this is the first step of the procedure. By iterating the procedure, we find at step k, $k \geq 2$, the interval whose endpoints are of the form

$$(a-c)\sum_{n=1}^{k} \frac{1}{2^{2n}}, \qquad (a-c)\left(\frac{1}{2} - \sum_{n=1}^{k-1} \frac{1}{2^{2n+1}}\right).$$

The conclusion follows by observing that letting k go to infinity we get $\frac{a-c}{4}$ in both cases.

4.2.2 The Bertrand model

This model differs from the previous one since it takes prices as strategies, and the quantity is required by the market according to the prices. Following the same ideas as in the Cournot model, it can be assumed that the total quantity required by the market, $q_1 + q_2$, be $q_1 + q_2 = \max\{a - (p_1 + p_2), 0\}$; furthermore, assuming that in case the prices made by the two firms agree they share the market, the payoff functions become the following:

$$u_1(p_1, p_2) = \begin{cases} (p_1 - c)(a - p_1) & \text{if } p_1 < p_2 \\ \frac{(p_1-c)(a-p_1)}{2} & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 > p_2, \end{cases}$$

$$u_2(p_1, p_2) = \begin{cases} (p_2 - c)(a - p_2) & \text{if } p_2 < p_1 \\ \frac{(p_2-c)(a-p_2)}{2} & \text{if } p_1 = p_2 \\ 0 & \text{if } p_2 > p_1, \end{cases}$$

Unfortunately, the only Nash equilibrium in this case is given by the choice of price c for both, giving payoff zero. The reason why another pair of prices, giving positive payoff, cannot be an equilibrium, is that by slighting lowering the price either firm can capture the entire market, making more profit. However the model can be slightly modified, taking into account the fact that the two firms cannot be considered completely symmetric; for instance a consumer accustomed to buy at one shop making the same price as another shop probably will not switch to the second if it lowers a bit the price. In order to try to take this into account, the model is modified by considering the required quantities q_1 and q_2, corrected in the following way:

$$q_1(p_1, p_2) = \max\{a - p_1 + bp_2, 0\}, \quad q_2(p_1, p_2) = \max\{a - p_2 + bp_1, 0\},$$

where $b > 0$ (small) is given the economical meaning of how the product of a firm is substitute of the product of the other one. Though these demand functions are still rather unrealistic since the firms, by coordinating their requests, could conceivably maintain very high prices and still having positive demand, yet the model provides interesting results. The two payoff functions become:

$$u_1(p_1, p_2) = \max\{a - p_1 + bp_2, 0\}(p_1 - c), \qquad u_2(p_1, p_2) = \max\{a - p_2 + bp_1, 0\}(p_2 - c),$$

and by using the usual methods we easily get:

$$\bar{p}_i = \frac{a + c}{2 + b}.$$

We leave to the reader to make some comparison with the previous result.

4.2.3 The Stackelberg model

In this model, already proposed in Exercise 2.8.24, there are still two competing firms, but now the idea is different in the following aspect. While the previous models suppose the firms acting simultaneously, here one firm, called the Leader, announces its strategy, and the other one, the Follower, acts taking for granted the announced strategy of the Leader. Thus, if the Leader announces q_1, the Follower will maximize its payoff function $q_2(a - (q_1 + q_2)) - cq_2$ (taking q_1 as parameter), and this provides

$$\bar{q}_2(q_1) = \frac{a - q_1 - c}{2}.$$

Now the Leader maximizes $u_1(q_1, \frac{a-q_1-c}{2})$ to finally get

$$\bar{q}_1 = \frac{a-c}{2}, \ \bar{q}_2 = \frac{a-c}{4}, \ u_1(\bar{q}_1, \bar{q}_2) = \frac{(a-c)^2}{8}, \ u_2(\bar{q}_1, \bar{q}_2) = \frac{(a-c)^2}{16}.$$

Observe that the Leader produces the same quantity as in the monopolistic case, while the Follower produces less than in the Cournot model. As a result, Leader earns more, Follower less, than in the Cournot case. As often, being the first to choose results in having an advantage, maybe against the first intuition. Finally, observe that the total amount of good in the market in the case of the Stackelberg model is $(3/4)(a-c)$, more than $(2/3)(a-c)$, the total amount in the Cournot case. And the Stackelberg price is better than in equilibrium price in the model of Cournot, and the consumers are thus better off in the system Leader-Follower. As a final remark, we observe that the Stackelberg model fits in the category of games with perfect information, it is a game of length two but not finite, in the sense that the players have a continuum of available strategies, rather than a finite number. However the method of backward induction applies, since the utility functions assume maxima on the strategy spaces.

4.3 Repeated Games

If the same players face the same game repeatedly, the strategic analysis can change in a complete manner. We start by considering the following example:

Example 4.3.1 Consider the following bimatrix:

$$\begin{pmatrix} (3,3) & (0,10) & (-2,-2) \\ (10,0) & (1,1) & (-1,-1) \\ (-2,-2) & (-1,-1) & (-2,-2) \end{pmatrix}.$$

Observe the following fact. Considering the first two rows/columns, we have a prisoner dilemma situation. For, domination brings to the outcome $(1,1)$, while the outcome $(3,3)$, better for both, cannot be implemented. Adding the third row/column does change nothing, since by strict domination we can come back to the previous outcome. Thus the unique Nash equilibrium, which is also the result of elimination of strictly dominated strategies, is the outcome $(1,1)$.
Suppose now the game is played N times.[7] Of course, we still have that the outcome $(1,1)$ is an equilibrium of the game, again obtained by backward induction and elimination of dominated strategies. However, maybe surprisingly, there are other Nash equilibria, and some of them provide interesting payoffs to the players. What I claim is that, for small a, the player can get, on average, the payment $3-a$ at each stage, if the horizon of the game (i.e. N) is sufficiently long.

The strategy bringing the players to this equilibrium is the same for both and is the following:

player one (two) plays the first row (column) at the first $N-k$ stages (k to be chosen later), and the second row (column) (i.e. the dominant strategy) in the last k stages, if the second (first) player uses the same strategy. Otherwise, if at one stage the second (first) deviates, from that stage on player one (two) plays the third row (column).

In more loose words, for some time the players collaborate, and in the final stages they use the dominating strategy. *This if both do the same.* If someone at a certain moment deviates form this policy, the other one *punishes* him by playing a strategy bringing a bad result to the opponent.

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[7] We shall introduce the positive parameters N, k, a (N, k integers, a small real number), later we shall see why they are introduced and how to choose them.

Let us see why this is a Nash equilibrium (here we make a choice for the parameters N and k). First of all, let us observe that, by playing the suggested strategy, the players get, on average,

$$\frac{(N-k)3+k1}{N}.$$

Then, observe that, for a player, the most convenient stage to deviate is the last stage in which the collaboration was requested, i.e. at the stage $N-k$.[8]

In this way, the player will get, on average, not more than:

$$\frac{(N-k-1)3+10+k(-1)}{N}.$$

Thus, the suggested strategy is a Nash equilibrium provided:

$$\frac{(N-k)3+k1}{N} \geq \frac{(N-k-1)3+10+k(-1)}{N}.$$

And this is true provided $k>3$. Now, observe that $\frac{(N-k)3+k1}{N}$ tends to 3, for all k, when N goes to infinity, and this shows that for all a, by taking N sufficiently large, the players get at least $3-a$ on average at each stage of the game.

A comment on this result. Clearly, it is an interesting result: it shows that a collaboration, even if dominated, can be based on rationality, *provided the game is repeated*. It seems we are finally out of the nightmare of the prisoner dilemma. However some remarks are in order. The first and more important is that the above equilibrium is one of the (many) equilibria of the game. Actually, the next theorem will show that the repetition of the game make the number of Nash equilibria grow explosively. This implies that a refinement of the concept of Nash equilibrium, in the repeated case, is necessary. Secondly, the Nash equilibrium proposed before has an evident weakness: it is based on a *threat* of the players, which is not completely credible: the first player, by choosing the third row, will punish also himself. Thus the second one could be interested in seeing if really the player will make such a choice.

Notwithstanding these objections, the above example has a remarkable meaning, also from a philosophical point of view: it says, among other things, that collaboration could be possible, but is fragile, and also that it *cannot* work at all stages: in the last ones, when the future is short, the players need to move to the dominant (egoistic) choice. The players in any case need this period (i.e. k cannot be zero), to have the time to punish the opponent, if necessary. It actually will not be necessary, at the condition of behave both correctly. Once again this example shows how the concept of Nash equilibrium is closely related to a *mutual* behavior of the players.

The above example is not only a curiosity. It can be generalized in several ways. In the next section we shall see a simple and informal way to do it.

4.3.1 The folk theorem

We shall suppose, in this version of the theorem, that the same game (called the stage game) is played infinite times, once at each (future) stage $\tau = 1, 2, \ldots$. We also

[8] In this way the number of stages in which the player is punished is the minimum possible.

suppose that at stage τ the players know which outcome has been selected at stage $\tau - 1$.[9] In this case it is worth to adapt the idea of strategy to the context. Let us explain this with the help of an example. Consider for instance the prisoner dilemma repeated twice. If we look at the associated game tree, see the Figure 4.1, a strategy for the first player must be a specification of what he does at the beginning of the game (stage zero, the root of the tree) and on each of the four nodes corresponding to all possible outcomes of the stage zero. However, since the stage game is always the same, we can assume that the player knows his choice, so that it is more convenient to ask him to specify his decision only at the nodes which are reachable according to his decision at the stage zero. Thus in this setting a strategy for a player could be $s = s(\tau)$, $\tau = 0, \dots,$ where for each τ $s(\tau)$ is a specification of moves of the stage game, depending from the past choices of the other players. Thus, as an example a typical strategy for this game could be: $s = (s(0), s(1))$ with $s(0)$ *do not confess*, $s(1)$ *do not confess if the other player did not confess at stage zero, otherwise confess.* I.e. we shall not specify, at stage 1, what the player should do in case he *confess* at stage zero.

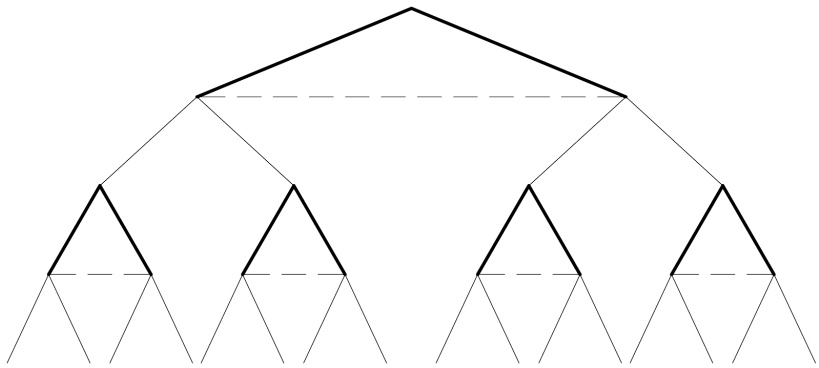

Fig. 4.1. The prisoner dilemma repeated twice.

A multistrategy, or a strategy profile, will as usual be a list of strategies, one for each player. Thus a multistrategy will determine a unique outcome of the game. We also need to specify utility functions. In order to do this, we introduce the discounting factor.

4.3.2 The discounting factor

The discounting factor δ is a value between 0 and 1, and is useful in building the utility functions for several reasons. Let us see it by means of two examples.

Example 4.3.2 Let us consider a constant inflative economic system, with yearly inflation coefficient α. Suppose a certain job is made by an industry, and that the total sum that will be paid for the job is M; suppose moreover that at the beginning

[9] This means that at each stage the players make simultaneous moves, which are then visible to both.

half of the due amount, i.e. $M/2$, will be paid, and that at the end of each year half of the remaining due amount is paid, i.e. at stage τ the paid sum is $\frac{M}{2^{\tau+1}}$. The amount of utility for the payment received at stage τ, will actually be multiplied by the factor $(1-\alpha^\tau)$, due to inflation. Thus the utility function will be of the form

$$u = \sum_{\tau=0}^{\infty}(1-\alpha)^\tau \frac{M}{2^{\tau+1}} = \frac{1}{2}\sum_{\tau=0}^{\infty}(\delta)^\tau M,$$

where $\delta = \frac{1-\alpha}{2}$ is the so called discount factor.

Example 4.3.3 Let's consider the situation of a stamp collectioner who must decide whether to by an expensive stamp. The stamp could possibly gain value each year; however, the paper used for the stamp can be attacked by a particular bacterium, so that after a certain unknown period the stamp could be irremediably damaged. If the collectioner waits until that time, the stamp has no more value. This is an example of indefinitely long process: in fact, the bacterium can act in such a long future to be more than life time. Thus when writing utility under the form of $\sum_0^\infty \delta^\tau u_i(\tau)$, while $u_i(\tau)$ denoted the value of the stamp at time τ, and can be increasing in τ, the factor δ subsumes the uncertainty about the fact that the game is still in play. Thus δ can measure at the same time impatience and uncertainty for future earnings, and also measure the subjective probability that at the stage τ the game actually will take place. Observe that δ close to zero means at the same time that you are impatient, and that you guess that the probability that the process ends quickly is high.

Thus the two examples above should justify the assumption of taking a utility function for player i as

$$u_i(s) = (1-\delta)\sum_{\tau=0}^{\infty}\delta^\tau u_i(s(\tau)),$$ [10]

where $u_i(s(\tau))$ is the stage-game payoff of player i at time τ through the pure multi-strategy $s(\tau)$.

In real situations, games are not repeated infinitely, but only a finite amount of times. The key point however is that the players usually don't know how many times the stage-game will be repeated. These games cannot be modelled as finite games, for instance it is impossible for the players to use backward induction, and thus an infinite horizon well represents the situation.

We can now see one theorem concerning these games. It sounds even paradoxical, but it has been a cornerstone of the theory of non cooperative games, and a starting point for further, much more refined results. In order to see it, we need to define a particular *value* attached to each player. We confine our analysis to the case when the stage game is given in form of a bimatrix. For the bimatrix game (A, B) the values are defined as:

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[10] The multiplicative factor $(1-\delta)$ obviously changes nothing in the analysis, it is used as a normalizing factor since in case $u_i(s(\tau)) = M$ constant, i.e. you get at each stage M, the total utility is M.

$$\underline{v}_1 = \min_j \max_i a_{ij}, \qquad \underline{v}_2 = \min_i \max_j b_{ij}.$$ [11]

The meaning of these values is clear: if a player, say player two, wants to *threat* player one, can guarantee to make him to earn at most $\underline{v}_1$. And conversely. Let us call *threat values* the above values of the players and *threat strategy* the strategy for the player making the other getting at most his threat value. For player i, we shall simply denote by $u_i(s)$ the payoff obtained by the implementation of strategy s. Finally, remember that for any $\delta \in (0,1)$ the following relation holds, for all $T_1 = 1, \dots$ and $T_2 = T_1, \dots, \infty$:

$$\sum_{\tau=T_1}^{T_2} \delta^\tau = \delta^{T_1} \frac{1 - \delta^{T_2+1}}{1-\delta}.$$

Theorem 4.3.1 The Folk Theorem *For every feasible payoff vector v such that $v_i > \underline{v}_i$ for all players i, there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$ there is a Nash equilibrium of the repeated game with discounting δ, with payoffs v.*

Proof. We prove the theorem in the particular case that v is the payoff of the outcome of a pure strategy in the basic game, i.e. there are $\mathbf{i}, \mathbf{j}$ such that $v = (a_{\mathbf{ij}}, b_{\mathbf{ij}})$. We consider the following strategy s for each player:[12]

player one (two) plays the strategy $\mathbf{i}$ *(*$\mathbf{j}$*) above at any stage, unless the opponent deviates at time t. In this case player one (two) plays the threat strategy form the stage $t+1$ ever.*

We prove now that the above is a Nash equilibrium of the game. We use in the argument player one, the situation being completely symmetric for player two.

If the player 1 deviates at time t, the other player will punish him from time $t+1$ onwards, letting him gain at most $\underline{v}_1$. On the other hand, at time t player i could gain at most $\max_{i,j} a_{ij}$. Denote by s_t the strategy of deviating at time t. We have for player one, by deviating at time t,

$$u_1(s_t) \le (1-\delta)\left(\sum_{\tau=0}^{t-1} \delta^\tau v_1 + \delta^t \max_{i,j} a_{ij} + \sum_{\tau=t+1}^{\infty} \delta^\tau \underline{v}_1\right) =$$
$$= (1-\delta^t) v_1 + (1-\delta)\delta^t \max_{i,j} a_{ij} + (\delta^{t+1})\underline{v}_1.$$

Instead, by using the proposed strategy:

$$u_1^s = (1-\delta)\sum_{\tau=0}^{\infty} \delta^\tau v_1 = v_1$$

By imposing $u_1(s_t) \le u_1(s)$:

$$(1-\delta^t) v_1 + \delta^t(1-\delta)\max_{i,j} a_{ij} + \delta^{t+1}\underline{v_1} \le v_1$$
$$\delta^t(1-\delta)\max_{i,j} a_{ij} + \delta^{t+1}\underline{v_1} \le \delta^t v_1$$
$$(1-\delta)\max_{i,j} a_{ij} + \delta\underline{v_1} \le v_1$$
$$\delta(\underline{v_i} - \max_{i,j} a_{ij}) \le v_1 - \max_{i,j} a_{ij}$$

[11] Be very careful, these are *not* the conservative values of the players.

[12] Observe the analogy with the equilibrium strategy of the Example 4.3.1.

Thus, if we set

$$0 < \underline{\delta_1} = \frac{\max_{i,j} a_{ij} - v_1}{\max_{i,j} a_{ij} - \underline{v_1}} < 1,$$

and performing the same calculation for the second player:

$$0 < \underline{\delta_2} = \frac{\max_{i,j} b_{ij} - v_1}{\max_{i,j} b_{ij} - \underline{v_1}} < 1,$$

the claim follows by choosing $\underline{\delta} = \max_{i=1,2} \underline{\delta_i}$. ■

4.4 Correlated equilibria

Let us consider the example of the battle of sexes:

$$\begin{pmatrix} (10,0) & (-5,-10) \\ (-10,-5) & (0,10) \end{pmatrix}.$$

There are two pure Nash equilibria. It is easy to verify (do it) that there is a third Nash equilibrium in mixed strategies, providing a payoff of -2 to both. It is a "fair" result, in the sense that it does not make preferences between the players (observe that the game is symmetric) but it is also clear that it is not a good idea to implement it, for instance if the game is played several times. It is more likely that the two players will play half times one of the equilibria, half times the other one. This means that they could toss a fair coin having in advance decided that head corresponds to one equilibrium, tail to the other one. This requires a form of communication/coordination between the players. But in this case something more interesting can be done. To describe this, let us consider a bit different game, which is very famous, since it is Aumann example in the paper introducing correlated equilibria:

Example 4.4.1

$$\begin{pmatrix} (6,6) & (2,7) \\ (7,2) & (0,0) \end{pmatrix}.$$

It is very easy to compute the (pure) Nash equilibria: they are got by playing (first row, second column) and (second row, first column), and provide payments $(2,7)$ and $(7,2)$ respectively to the players. A typical situation where non uniqueness is bothering. Moreover, it is easy to see (verify!) that there is a third equilibrium, in mixed strategies, providing an amount of $\frac{14}{3}$ to both (in this sense, it is a more fair result). And in this case, the result is not as bothering as in the battle of sexes. But the question is: could they do better? Suppose they decide to play with equal probability the outcomes $(6,6)$, $(2,7)$ and $(7,2)$. An easy calculation shows that they will get the expected payoff of 5. Not only the outcome is fair, in the sense that it does not give an advantage to one of the players, but it is also better than the equilibrium in mixed strategies. However, we must remember that the agreements between players cannot be binding, in the non cooperative theory. In other words, if an authority suggests the outcome $(6,6)$ to the players, they will not follow the advise, since the outcome is not a Nash equilibrium. The brilliant Aumann's idea is to discover a simple way to make self-enforcing the agreement! Let us see how. The arbitrator can say to the players: if you agree to play the three outcomes with equal probability, then I will use a random device to select the outcome, and I will tell you what to play, but *privately*. In other words, John knows what to do, but does not know what I will say to Alicia, and conversely. Does this system work? Yes, indeed, since the players will not have any incentive tio deviate to a strategy different from that one suggested. Let us see why. Suppose I say to John (the first player): play bottom. He know for sure that I will tell to Alicia, play left, and thus he does not have interest to deviate since the outcome is a Nash equilibrium. Suppose instead I say to John, play top. At this point, he knows that there is equal probability that I have said left or right to Alicia, and his expected payoff will be 4. If he decides to deviate, he will receive 3.5, so that deviating for him is not convenient! Of course it is necessary to make the same check for Alicia, but as it is easy to see, the same argument applies to her as well. Thus, the agreement is self enforcing, provided the information to the players is not *complete*.

Thus, the idea of Aumann is to define the new concept of correlated equilibrium, in the following fashion. I will do it for two players, just for easy notation: the idea can be easily extended to a (finite but) arbitrary number of players.

Suppose we are given a bimatrix $(A, B) = (a_{ij}, b_{ij})$, $i = 1, \dots, n$, $j = 1, \dots, m$, representing a finite game (non zero sum, two players). Let $I = \{1, \dots, n\}$, $J = \{1, \dots, m\}$ and $X = I \times J$. A *correlated equilibrium* is a probability distribution $p = (p_{ij})$ on X such that, for all $\bar{i} \in I$,

$$\sum_{j=1}^{m} p_{\bar{i}j} a_{\bar{i}j} \geq \sum_{j=1}^{m} p_{\bar{i}j} a_{ij} \qquad \forall i \in I, \tag{4.1}$$

and such that, for all $\bar{j} \in J$

$$\sum_{i=1}^{n} p_{i\bar{j}} b_{i\bar{j}} \geq \sum_{i=1}^{n} p_{i\bar{j}} b_{ij} \qquad \forall j \in J. \tag{4.2}$$

Maybe not very easy to understand at a first glance, but it is exactly what we have done in the example above, if we notice the simple fact that the quantity:

$$\sum_{j=1}^{m} \frac{p_{\bar{i}j}}{\sum_{j=1}^{m} p_{\bar{i}j}} a_{\bar{i}j},$$

is exactly the expected payoff of the player one after receiving the information to play $\bar{i}$. Thus the equations (4.1) and (4.2), called *incentive constraint inequalities*, make only disappear a positive multiplicative coefficient.[13]

It follows that a correlated equilibrium is a vector satisfying a given number of linear inequalities (plus a normalization condition). So that, the set of the correlated equilibria is a convex compact set, in some Euclidean space. Is it nonempty? The answer is easy: every Nash equilibrium is a correlated equilibrium, in the following sense. Suppose $(p, q) = ((p_1, \dots, p_n), (q_1, \dots, q_m))$ is a Nash equilibrium. Then $(p_i q_j)$ is a correlated equilibrium. To see this, let us verify that the incentive constraint

[13] This is true for $\bar{i}$ such that $\sum_{j=1}^{m} p_{\bar{i}j} a_{\bar{i}} > 0$. On the other hand, if the row $\bar{i}$ is played with null probability (4.1) is obvious and the meaning given to the definition remains the same.

inequalities are fulfilled, the fact that $(p_i q_j)$ belongs to the simplex is trivial. Let us write the conditions for the first player. We have to prove that

$$\sum_{j=1}^{m} p_{\bar{i}} q_j a_{\bar{i}j} \geq \sum_{j=1}^{m} p_{\bar{i}q_j} a_{ij} \qquad \forall i \in I, . \tag{4.3}$$

This is obvious if $p_{\bar{i}} = 0$. Otherwise we need to show that, if $p_{\bar{i}} > 0$, then

$$\sum_{j=1}^{m} q_j a_{\bar{i}j} \geq \sum_{j=1}^{m} q_j a_{ij} \qquad \forall i \in I, . \tag{4.4}$$

But this is exactly the definition of Nash equilibrium: if a pure strategy is played with positive probability at an equilibrium, it must be a best reaction to the strategy of the other player.
Thus, since finite bimatrix games have a Nash equilibrium, it follows that the set of the correlated equilibria is always nonempty. However, it is natural to ask the question if it is possible to give an independent proof, using classical convexity arguments (rather than a fixed point argument, necessary to show existence of Nash equilibria). The answer is positive, and moreover it can be noticed that finding the set of the correlated equilibria is in principle simpler than computing the Nash equilibria, since the problem can be put under the form of a linear programming problem. Even more, we could do one more step and, since usually the set is big, we can add one more condition, such that for instance to maximize the total payoff of the players.

Just as an example, let us write down the conditions characterizing the correlated equilibria in Example 4.4.1:

$$\begin{cases} 6x_1 + 2x_2 \geq 7x_1 \\ 7x_3 \geq 6x_3 + 2x_4 \\ 6x_1 + 2x_3 \geq 7x_1 \\ 7x_2 \geq 6x_2 + 2x_4 \\ x_1 + x_2 + x_3 + x_4 = 1 \\ x_i \geq 0 \qquad i = 1, \dots, 4. \end{cases} .$$

In case we want to add a further condition, for instance to maximize the sum of the payoffs, we finally get:

$$\begin{cases} \text{Maximize } 4x_1 + 3x_2 + 3x_3 \\ \text{subject to :} \\ -x_1 + 2x_2 \geq 0 \\ x_3 - 2x_4 \geq 0 \\ -x_1 + 2x_3 \geq 0 \\ x_2 - 2x_4 \geq 0 \\ x_1 + x_2 + x_3 + x_4 = 1 \end{cases} .$$

4.5 Exercises

Exercise 4.5.1 Find the Nash and correlated equilibria in pure strategies for the Prisoner Dilemma.

Solution The Prisoner Dilemma is described by a bimatrix like the following one:

$$\begin{pmatrix} (5,5) & (0,7) \\ (7,0) & (1,1) \end{pmatrix}$$

By eliminating strictly dominated strategies we find that the only Nash equilibrium is given by $(5,5)$. For this reason this is also the unique correlated equilibrium and hence the unique equilibrium in mixed strategies.

Exercise 4.5.2 Find the Nash equilibria in pure strategies for the Battle of Sexes.

Solution The Battle of Sexes is described by a bimatrix like the following one:

$$\begin{pmatrix} (10,0) & (-5,-10) \\ (-10,-5) & (0,10) \end{pmatrix}$$

there are no dominated strategies. The Nash equilibria are given by $(10,0)$ and $(0,10)$. To find the equilibria in mixed strategies we suppose player I plays $(p, 1-p)$ and II plays $(q, 1-q)$. Then

$$f(p,q) = 5pq + 7(1-p)q + (1-p)(1-q) = p(-q-1) + 6q + 1$$

and the best reply function for the first player is

$$BR_I(p,q) = \begin{cases} p = 1 & q > \frac{1}{5} \\ [0,1] & q = \frac{1}{5} \\ p = 0 & q < \frac{1}{5}. \end{cases} \quad .$$

$$g(p,q) = 5pq + 7p(1-q) + (1-p)(1-q)$$

and the best reply function for the second player is

$$BR_{II}(p,q) = \begin{cases} q = 1 & p > \frac{3}{5} \\ [0,1] & p = \frac{3}{5} \\ q = 0 & p < \frac{3}{5}. \end{cases} \quad .$$

This two best reply functions have three intersection points: $(0,0)$ and $(1,1)$ (which correspond to the equilibria in pure strategies $(10,0)$ and $(0,10)$) and the equilibrium in mixed strategies $\left(\left(\frac{3}{5}, \frac{2}{5}\right), \left(\frac{1}{5}, \frac{4}{5}\right)\right)$.

Exercise 4.5.3 Given the following game

$$\begin{pmatrix} (2,1) & (-3,-1) \\ (-1,0) & (0,1) \end{pmatrix},$$

1. find the Nash equilibria in pure strategies
2. find the Nash equilibria in mixed strategies.

Solution

1. The equilibria in pure strategies are $(2,1)$ and $(0,1)$

2. assuming I plays the mixed strategy $(p, 1-p)$ and II $(q, 1-q)$ we have

$$f(p,q) = 3p(2q-1) - q$$

then $q = \frac{1}{2}$ and

$$g(p,q) = q(3p-1) - 2p + 1$$

from which we get $p = \frac{1}{3}$, then the equilibrium in mixed strategies is $\left(\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$.

Exercise 4.5.4 Find the equilibria in pure and mixed strategies of the following game

$$\begin{pmatrix} (2,4) & (3,3) & (8,0) \\ (1,1) & (4,2) & (5,0) \end{pmatrix},$$

Find, if there is, a correlated equilibrium of the form:

$$\begin{pmatrix} x & 0 & 0 \\ y & z & 0 \end{pmatrix},$$

Solution We start observing that the third column is strictly dominated. The equilibria in pure strategies are $(2,4)$ and $(4,2)$. To find the equilibria in mixes strategies we assume $(p, 1-p)$ as the strategy for the first player and $(q, 1-q)$ for the second one.
From

$$f(p,q) = p(2q-1) - 3q + 4$$

we have $q = \frac{1}{2}$. From

$$g(p,q) = q(2p-1) + p + 2$$

we have $p = \frac{1}{2}$. The equilibrium in mixed strategies is

$$\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}, 0\right)\right)$$

To find a correlated equilibrium of the form

$$\begin{pmatrix} x & 0 & 0 \\ y & z & 0 \end{pmatrix},$$

it is immediate that

$$\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix},$$

is a correlated equilibrium (as a convex combination of two Nash equilibria).
To find a correlated equilibrium of this form, but with $x, y, z > 0$, we write the incentive constraints

$$\begin{cases} 2x \geq x \\ y + 4z \geq 2y + 3z \\ 4x + y \geq 3x + 2y \\ 2z \geq z \end{cases}$$

with $x + y + z = 1$; a possible correlated equilibrium we obtain is

$$\begin{pmatrix} 1/3 & 0 & 0 \\ 1/3 & 1/3 & 0 \end{pmatrix},$$

Exercise 4.5.5 Given the following game, in which player III chooses the matrix, I the row and II the column:

$$\begin{pmatrix} (1,0,0) & (7,3,1) \\ (1,10,1) & (4,2,0) \end{pmatrix}, \begin{pmatrix} (2,4,5) & (3,3,2) \\ (1,1,3) & (4,2,1) \end{pmatrix},$$

1. prove that in every correlated equilibrium player III never chooses the first matrix
2. find the Nash equilibria in pure strategies
3. find the Nash equilibria in mixed strategies.

Solution

1. The first matrix is strictly dominated by the second one (for the third player), then it will never be played at the equilibrium
2. The Nash equilibria in pure strategies are $(2,4,5)$ and $(4,2,1)$
3. Another equilibrium in mixed strategies is $\left(\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right),(0,1)\right)$.

Exercise 4.5.6 Given the bimatrix:

$$\begin{pmatrix} (a,1) & (1,0) & (2,b) \\ (4,8) & (3,4) & (4,1) \\ (1,0) & (0,2) & (8,2) \end{pmatrix},$$

$a, b \in \mathbb{R}$:

1. find the equilibria in pure strategies
2. for which a, b the strategy $(1/3, 1/3, 1/3)$ of the first player is part of a Nash equilibrium and find all Nash equilibria.

Solution

1. $(8,2)$ is an equilibrium in pure strategies $\forall a, b \in \mathbb{R}$;
   if $a \geq 4$ and $b \leq 1$, $(a,1)$ is equilibrium; if $a \leq 4$, $(4,8)$ is equilibrium $\forall b \in \mathbb{R}$
2. We evaluate the expected values for the second player, given the strategy $(1/3, 1/3, 1/3)$ of the first player:
   I column: 3
   II column: 2
   III column: $\frac{b+3}{3}$.
   There are no Nash equilibria with the second player playing a pure strategy, in fact if, for example, player II plays $(1,0,0)$, player I should be indifferent between the three rows, then $\frac{a}{3} = \frac{4}{3} = \frac{1}{3}$, which is absurd (similarly for the other pure strategies). Next, the second column is strictly dominated, then if $b+3=9$, i.e. $b=6$, a convex combination of the first and of the second column is an equilibrium, for example $(q, 0, 1-q)$. I is indifferent between the 3 rows, this implies $aq + 2(1-q) = 4q + 4(1-q) = q + 8(1-q)$ from which we get $q = \frac{4}{7}$ and $a = \frac{11}{2}$. The Nash equilibrium is indeed $\left(\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),\left(\frac{4}{7},0,\frac{3}{7}\right)\right)$ when $a = \frac{11}{2}$ and $b = 6$.

Exercise 4.5.7 Given the bimatrix:

$$\begin{pmatrix} (1,0) & (0,3) \\ (0,2) & (1,0) \end{pmatrix},$$

1. find a Nash equilibrium in mixed strategies
2. find the maxmin strategy for the two players.

Solution

1. The Nash equilibrium is $\left(\left(\frac{2}{5},\frac{3}{5}\right),\left(\frac{1}{2},\frac{1}{2}\right)\right)$ from which the two players can obtain $\left(\frac{1}{2},\frac{6}{5}\right)$
2. We have

$$\min_q f(p,q)=\min_q(2pq-p-q+1)=\begin{cases} p & p<\frac{1}{2}\\ \frac{1}{2} & p=\frac{1}{2}\\ 1-p & p>\frac{1}{2}\end{cases}$$

and

$$\min_q g(p,q)=\begin{cases} 3-3q & q>\frac{3}{5}\\ \frac{6}{5} & q=\frac{3}{5}\\ 2q & q<\frac{3}{5}\end{cases}$$

The minmax strategy is $\left(\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{3}{5},\frac{2}{5}\right)\right)$ from which the two players can obtain $\left(\frac{1}{2},\frac{6}{5}\right)$: the same utilities as in the Nash equilibrium: it is a non profitable game.

Exercise 4.5.8 Given the following game, in which player III chooses the matrix, I the row and II the column:

$$\begin{pmatrix}(1,3,1) & (1,2,0)\\ (3,2,0) & (0,3,1)\\ (0,1,1) & (3,2,0)\end{pmatrix},\begin{pmatrix}(1,1,0) & (1,0,1)\\ (2,1,1) & (3,2,0)\\ (3,0,0) & (0,3,1)\end{pmatrix}$$

1. find the equilibria in pure strategies
2. find the equilibria in mixed strategies.

Solution

1. There are no equilibria in pure strategies
2. The first row is strictly dominated by the mixed strategy given by 1/2 the second row and 1/2 the third one. Now the first column is strictly dominated by the second one.
   The reduced game becomes

$$\begin{pmatrix}(0,1) & (3,0)\\ (3,0) & (0,1)\end{pmatrix},$$

where the first player chooses a row and the third one chooses a column.
To find the equilibria in mixes strategies we assume $(p,1-p)$ as the strategy for the first player in this new bimatrix and $(q,1-q)$ for the third one. Then

$$f(p,q)=-3p(2q-1)+3q$$

from which we get $q=\frac{1}{2}$ and

$$g(p,q)=q(2p-1)-p+1$$

providing $p=\frac{1}{2}$. The Nash equilibrium is $\left(\left(0,\frac{1}{2},\frac{1}{2}\right),(0,1),\left(\frac{1}{2},\frac{1}{2}\right)\right)$.

Exercise 4.5.9 Define two games as being equivalent if they have the same Nash equilibria. Prove that, given a bimatrix game (A,B) an equivalent game can be obtained by subtracting one row (column, respectively) to all other rows (columns respectively) of A (B, respectively).

Exercise 4.5.10 Find the Nash equilibria, in pure strategies, of the following bimatrix game:

$$A = \begin{pmatrix} (1,1) & (5,3) & (6,8) \\ (9,8) & (1,12) & (1,3) \end{pmatrix}.$$

Exercise 4.5.11 Given the bimatrix game:

$$A = \begin{pmatrix} (a,a) & (b,d) \\ (d,b) & (c,c) \end{pmatrix},$$

with $a < d$, $b < c$, $c < a$, find all Nash and mixed equilibria.

Exercise 4.5.12 Consider the ultimatum game: player one is offered 100 euros, and must make an offer of at least one to player two. If the second refuses, the money is lost.

1. find the backward induction solution
2. find the dimension of the associated bimatrix
3. find all pure Nash equilibria.

Exercise 4.5.13 Let A be a bimatrix 2×2 non profitable game with unique Nash equilibrium in pure strategies. Then the maxmin and Nash strategies for the players do agree.

Exercise 4.5.14 Given the bimatrix

$$A = \begin{pmatrix} (3,3) & (6,1) & (1,3) \\ (1,6) & (1,1) & (6,4) \\ (2,1) & (4,6) & (5,5) \end{pmatrix},$$

find the (pure) Nash equilibria and the (pure) conservative strategies. Compare the payoffs.

Exercise 4.5.15 Consider the following bimatrix:

$$A = \begin{pmatrix} (10,10) & (0,11) & (0,0) \\ (11,0) & (1,1) & (0,0) \\ (0,0) & (0,0) & (-1,1) \end{pmatrix}.$$

Given $\varepsilon > 0$, find how many times the game should be played in order that there exists a Nash equilibrium providing at least $10 - \varepsilon$, on average, to both players.

Exercise 4.5.16 Two workers can decide whether to work a lot h or a little l, and this not observable. If they both contribute h, the total earnings is 156, otherwise 96 and 6. The salary is equal to both, and utility is given by the salary in case of l and the salary minus 39 in case of h. Write the bimatrix, with l in the first row/column, and find the Nash and mixed equilibria. Denoted by

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

a correlated equilibrium of the game, say if

$$\begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{375}{1000} \\ \frac{375}{1000} & \frac{250}{1000} \end{pmatrix}$$

are correlated equilibria of the game.

Exercise 4.5.17 Find all classes of equivalent two by two bimatrix games without dominated strategies, and discuss associated Nash and correlated equilibria.

Exercise 4.5.18 Find all Nash, mixed and correlated equilibria of the following bimatrix:

$$A = \begin{pmatrix} (1,0) & (2,0) \\ (1,2) & (2,2) \end{pmatrix}.$$

Explain the result.

Exercise 4.5.19 Given the bimatrix

$$A = \begin{pmatrix} (2,2) & (0,3) & (0,0) \\ (1,1) & (-1,-1) & (1,1) \\ (0,0) & (0,3) & (2,2) \end{pmatrix},$$

1. Is the probability distribution: play (first row,first column) and (third row,third column) with probability $\frac{1}{2}$ each a correlated equilibrium?
2. Find, if there exists, a correlated equilibrium assigning positive probability to both the above strategy profiles.

Exercise 4.5.20 Given a bimatrix A and a correlated equilibrium (p_{ij}) for it, prove that (p_{ij}) is a mixed Nash equilibrium if and only if the following holds, for all i, j, k, l:

$$p_{ij}p_{kl} = p_{il}p_{kj}.$$

Exercise 4.5.21 Given the bimatrix

$$A = \begin{pmatrix} (1,1) & (1,1) \\ (0,1) & (2,2) \end{pmatrix},$$

apply the indifference principle to find Nash equilibria. Do the same by drawing the best reaction multifunctions and compare the results.

Exercise 4.5.22 Prove that the Nash equilibria of a non trivial game[14] are in the boundary of the polytope of the correlated equilibria, in its relative boundary if it is full dimensional (i.e. a Nash equilibrium must verify at least one of the above (non trivial) conditions with equality).

[14] A game is non trivial if there is at least one player such that she is able to have two different outcomes, for at least one fixed choice of the other players. In the two players case, and supposing that the player is the first, there are $x, z \in X$ and $y \in Y$ such that $f(x, y) \neq f(z, y)$.

5

Cooperative Games

So far, we have considered situations where there are not possible binding commitments among players: each one can announce a strategy, but then he is not obliged to follow it when the game is played. However, it is not difficult to imagine situations where the players are forced to maintain pacts. In this case it is useful to develop a theory where binding commitments are allowed. Here we develop some fundamentals about this.

5.1 The model

The setting is the following: we have a (finite) set, denoted by N, which represents the set of the players. A subset of N, say A, is called *coalition.* The number $|A|$ of the players in A will be denoted by a (thus n is the number of the players). Denote also by 2^N the family of the coalitions of N.

Definition 5.1.1 *A side payment cooperative game is a function*

$$v : 2^N \to \mathbb{R},$$

such that $v(\emptyset) = 0$.

The idea behind this definition is the following. The players have the possibility to form coalitions. Each coalition A is able to get for itself (once it is formed) the amount $v(A)$. The amount $v(A)$ can be distributed among the members of A, in any way. The condition $v(\emptyset) = 0$ is a sort of normalization condition.

Example 5.1.1 I propose to my three children to give 300 Euros to whom will get at least two votes by them. They can make agreements, I will control that they will be respected. If no one gets two votes (this means that each one votes for himself, quite possibly), I will keep the 300 Euros for myself. In this case $v(A) = 300$ if $|A| \geq 2$, zero otherwise. Observe, it would be possible to put $v(A) = 1$ if $|A| \geq 2$, zero otherwise as well, the two games are equivalent.

Example 5.1.2 *(Buyers and sellers)*

- There are one seller and two potential buyers for an important, indivisible good. Let us agree that the player one, the seller, evaluates the good a. Players two and three evaluate it b and c, respectively. We assume that $b \leq c$ (this is not a real assumption) and that $a < b$ (this is just to have a real three player game). The following is the associated game:

$$v(\{1\}) = a, v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0,\ v(\{1,2\}) = b, v(\{1,3\}) = c,\ v(N) = c.$$

- There are two sellers and one potential buyer for an important, indivisible good. The following is the associated game:

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,2\}) = 0, v(\{1,3\}) = v(\{2,3\}) = v(N) = 1.$$

The use of the term side payment (transferable utility is used too) highlights the fact that the amount $v(A)$ can be freely divided among the members of A, without restrictions. There are situations where this is not possible, thus a more general definition is in order. The value $v(A)$ is substituted by a subset $V(A)$ of $\mathbb{R}^A$, and a vector in this set has the meaning of a possible division of utilities among the members of A. In the case of TU games,

$$V(A) = \{(x_i)_{i \in A} : \sum_{i \in A} x_i \leq v(A)\}.$$

When the set N of the players is fixed, the set $\mathcal{G}$ of all cooperative games among them can be identified with $\mathbb{R}^{2^n-1}$, in the following way: write a vector listing in any order the $2^n - 1$ coalitions. Then a vector $(v_1, v_2, \dots, v^{2^n-1})$ of $\mathbb{R}^{2^n-1}$ is a game in the sense that v_1 represents the value assigned the first coalition of the above list, and so on.
For instance, let fix $N = \{1, 2, 3\}$ and the following list of coalitions:

$$\{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{N\}\},$$

and the vector $(0, 0, 0, 1, 1, 1, 1)$. This a game equivalent to that one of Example 5.1.1.

It follows that the set $\mathcal{G}$ is a vector space of dimension $2^n - 1$. Can we produce a basis for it? Of course we can, it is for instance possible to consider the family $\{e_A : A \subset N\}$ of games:

$$e_A(T) = \begin{cases} 1 & \text{if} \quad A = T \\ 0 & \text{otherwise} \end{cases}. \tag{5.1}$$

This corresponds to consider the canonical basis on $\mathbb{R}^{2^n-1}$, but it is not a family of interesting games. It is then possible to consider a more interesting basis for $\mathcal{G}$. Consider the family $\{u_A : A \subset N\}$ of the *unanimity games*, defined in the following way:

$$u_A(T) = \begin{cases} 1 & \text{if} \quad A \subset T \\ 0 & \text{otherwise} \end{cases}. \tag{5.2}$$

To prove that the family of unanimity games is a basis for $\mathcal{G}$ it is enough to show that every $v \in \mathcal{G}$ can be written as a linear combination of the games u_A. Thus, let us prove that any game v can be written as

$$v = \sum_{T \subset N} c_T u_T,$$

for a suitable choice of the coefficient c_T. The constants c_A can be inductively defined in the following way: let $c_{\{i\}} = v(\{i\})$ and, for $A \subset N$, $a \geq 2$ define

$$c_A = v(A) - \sum_{T \neq A, T \subset A} c_T. \tag{5.3}$$

Let us verify that the above choice of constants c_A does the job. For every coalition A:

$$\sum_{T \subset N} c_T u_T(A) = \sum_{T \subset A} c_T = c_A + \sum_{T \neq A, T \subset A} c_T = v(A). \tag{5.4}$$

There are subsets of $\mathcal{G}$ representing interesting classes of games: one of them is the class $\mathcal{SG}$ of the superadditive games. A game is said to be superadditive if:

$$v(A \cup B) \geq v(A) + v(B),$$

provided the two coalitions A, B are disjoint. In other words, staying together is better than being separated: this is quite reasonable in many situations, but it is not necessary to assume it as a part of the definition of cooperative game.
Another interesting subclass of games is singled out in the following definition.

Definition 5.1.2 *A game $v \in G$ is called* simple *provided v is valued on $\{0, 1\}$, $A \subset C$ implies $v(A) \leq v(C)$ and $v(N) = 1$.*

Typically, the simple games are majority games, and coalitions A for which $v(A) = 1$ are the *winning* coalitions.

It is time to introduce the idea of solution for a cooperative game. In general, a *solution vector* for a game v will be a vector $(x_1, \dots, x_n)$, where x_i represents the amount assigned to the player i.[1] A *solution concept* for the game v is a (possibly empty) set of solution vectors of the game. Finally, a solution on $\mathcal{G}$ (or subset of it) is a multifunction defined on $\mathcal{G}$ (or subset of it) and valued in $\mathbb{R}^n$.

What type of requirements is reasonable to ask to a solution? First of all, let us mention that there are very many solution concepts for cooperative games. This is due to the fact that a cooperative game is a complex model and, differently from the non cooperative case, it is not possible to define a solution concept which provides a reasonable solution for all situation a game would like to model. But must not be considered a weakness of the theory: each solution provides important information on the game. With this premise, we can argue that the following are two minimal conditions to be required to any solution x:

1. $x_i \geq v(\{i\})$ for all i

[1] This very loose definition is given only to emphasize the fact that here we look at the final outcome only in terms of utilities. No strategy argument is present in this theory, and this partly explains why it is less considered in economics.

2. $\sum_{i=1}^{n} x_i = v(N)$.

The first condition has a simple meaning: if a solution assigns to a player less than he is able to do by himself, clearly this player will not participate to the game, so that actually the set of the players is not the set N. The second condition as usual can be split in two inequalities: $\sum_{i=1}^{n} x_i \leq v(N)$, means that what is proposed is actually feasible (if the grand coalition N will form $v(N)$ is the quantity available for the players), $\sum_{i=1}^{n} x_i \geq v(N)$ is an efficiency condition, since the whole amount $v(N)$ is distributed. This makes a big difference with the non cooperative case. Thus, the first restriction to be imposed to a solution vector is of being a imputation.

Definition 5.1.3 *Given a game v, we call* imputation *any vector x fulfilling the conditions 1., 2. above.*

We shall denote by $I(v)$ the set of the imputations of the game v, and we assume so forth that $I(v) \neq \emptyset$, when needed, without mentioning it. It is clear that an outcome of the game must be inside the imputations, but this is not enough to characterize a solution, or a solution set, since it is in general too large. As already mentioned, the solutions proposed in the literature are several. Here, we shall analyze three different concepts, called the core, the Shapley value and the nucleolus. We shall also spend few words on the so called indices of power.

5.2 The core

Definition 5.2.1 *Let $v : 2^N \to \mathbb{R}$ be a side payment game. The* core *of the game, denoted by $C(v)$, is the set:*

$$C(v) = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = v(N) \quad \wedge \quad \sum_{i \in S} x_i \geq v(S) \quad \forall S \subset N \right\}.$$

Thus the idea behind the definition of the core is to enforce what we did when defining the set of the imputations: there, we argued that only distributions of utilities such that neither any single player nor the grand coalition N can object should be taken into account. The core throws away all imputations rejected by at least one coalition. Observe that the definition of core is not particularly meaningful for a two player game: all imputations belong to the core, and conversely.

Example 5.2.1 The core in the game of Example 5.1.2 is of the form:

$$C(v) = \{(x, 0, c - x) : \quad b \leq x \leq c\}.$$

Example 5.2.2 Let v be the following three player game: $v(S) = 1$ if $|S| \geq 2$, otherwise $v(S) = 0$ (it is the game of Example 5.1.1). Let us see that the core of v is empty. Suppose (x_1, x_2, x_3) is in the core. Looking at the coalitions made by two members, it must be that:

$$x_1 + x_2 \geq 1$$

$$x_1 + x_3 \geq 1$$

$$x_2 + x_3 \geq 1.$$

On the other hand,

$$x_1 + x_2 + x_3 \geq 1.$$

Adding up the first three inequalities we get

$$2x_1 + 2x_2 + 2x_3 \geq 3,$$

contradicting the last one.

Why the core is empty in the above example? It is apparent that the coalitions of two players are too strong: they can all get the whole booty. To have non-emptiness, it is necessary to impose conditions ensuring that the power of intermediate coalitions be not too strong with respect to the gran coalition N.

The result of the Example 5.1.2 is not surprising at all. The good will be sold to the guy evaluating it more, to a price which can vary from the price offered by the person evaluating it less to the maximum possible price. This is quite reasonable: the price cannot be less that b, otherwise the second player could offer more. On the other hand, it cannot be more than c, since the third player does not buy a good for a price higher than the value he assigns to the good itself. Not completely satisfactory, as an answer: we would be happy to have a more precise information. There are other solution concepts suggesting a single vector in this case (precisely, the price will be $(b+c)/2$, for the so called *nucleolus*, see later). Observe that the second player, though getting nothing and buying nothing, has a role in the game, since his presence forces the third player to offer at least b. Without him, the third could buy the good at any amount greater than a.

The Example 5.2.2 shows that the core of a game can be empty. Thus it is of great interest to find out conditions under which we can assure non emptiness of the core. An elegant result holds for the simple games.

Definition 5.2.2 *In a game v, a player i is a* veto player *if $v(A) = 0$ for all A such that $i \notin A$.*

Theorem 5.2.1 *Let v be a simple game. Then $C(v) \neq \emptyset$ if and only if there is at least one veto player.*

Proof. If there is no veto player, then for every i there is A_i such that $i \notin A_i$ and $v(A_i) = 1$. Suppose $(x_1, \dots, x_n)$ belongs to the core. It follows that

$$\sum_{j \neq i} x_j \geq \sum_{j \in A_i} x_j = 1,$$

for all $i = 1, \dots, n$. Summing up the above inequalities, as i runs from 1 to n, provides:

$$(n-1) \sum_{j=1}^{n} x_j = n,$$

a contradiction since $\sum_{j=1}^{n} x_j = 1$. Conversely, we leave the reader to show that any imputation assigning zero to the non-veto players is in the core. ■

About the geometrical/topological structure of the core, it is easy to see that, for every v, $C(v)$ is a (possibly empty) convex closed bounded set. More precisely, the vectors in the core can be characterized as those vectors fulfilling a list of linear inequalities. This leads again to the idea of trying to characterize the core as the solution set of a particular linear programming problem, and then to look at its dual problem, to get extra information. This is what we are going to illustrate.

Consider the following linear programming problem:

$$\text{minimize} \sum_{i=1}^{n} x_i, \tag{5.5}$$

$$\text{subject to} \sum_{i \in S} x_i \geq v(S) \ \text{ for all } S \subset N. \tag{5.6}$$

Clearly, it is feasible and has at least a solution (why?). Call $\bar{x}$ one of these solutions, and look at $\sum_{i=1}^{n} \bar{x}_i$. If it is $\sum_{i=1}^{n} \bar{x}_i \leq v(N)$, then $\bar{x}$ lies in the core. And conversely. Thus $C(v) \neq \emptyset$ if and only if the linear programming problem (5.5),(5.6) has value $\bar{v}$ such that $\bar{v} \leq v(N)$, and when it is nonempty the elements of the core are exactly the solutions of the above LP problem.

Just to familiarize with this, let us write the above linear programming problem for the three player game.

$$\begin{aligned}
&\text{minimize}\, x_1 + x_2 + x_3, \\
&\text{subject to}\, x_i \geq v(\{i\}) : \ i = 1,2,3 \quad \text{and} \\
&x_1 + x_2 \geq v(\{1,2\}) \\
&x_1 + x_3 \geq v(\{1,3\}) \\
&x_2 + x_3 \geq v(\{2,3\}) \\
&x_1 + x_2 + x_3 \geq v(N).
\end{aligned}$$

In matrix form:

$$\begin{cases} \min\langle c, x\rangle \text{ such that} \\ Ax \geq b \end{cases},$$

where c, A, b are the following objects:

$$c = (1,1,1), \qquad b = (v(\{1\}), v(\{2\}), v(\{3\}), v(\{1,2\}), v(\{1,3\}), v(\{2,3\}), v(N))$$

and A is the following 7×3 matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

The matrix A has dimension $(2^n - 1) \times n$, to each column is associated a player, and each row is associated to a coalition (containing the players corresponding to the 1's in that row). The dual variable has $2^n - 1$ components, and a good idea is to use the letter S, denoting a coalition, for its index. Thus, in our example a generic dual variable is denoted by $(\lambda_{\{1\}}, \lambda_{\{2\}}, \lambda_{\{3\}}, \lambda_{\{1,2\}}, \lambda_{\{1,3\}}, \lambda_{\{2,3\}}, \lambda_N)$ and the dual problem (see Theorem 6.3.1) becomes:

$$\max \big(\lambda_{\{1\}} v(\{1\}) + \lambda_{\{2\}} v(\{2\}) + \lambda_{\{3\}} v(\{3\}) +$$

$$+\lambda_{\{1,2\}} v(\{1,2\}) + \lambda_{\{1,3\}} v(\{1,3\}) + \lambda_{\{2,3\}} v(\{2,3\}) + \lambda_N v(N)\big),$$

$$\text{subject to } \lambda_S \geq 0 \ \forall S \quad \text{and}$$

$$\lambda_{\{1\}} + \lambda_{\{1,2\}} + \lambda_{\{1,3\}} + \lambda_N = 1,$$

$$\lambda_{\{2\}} + \lambda_{\{1,2\}} + \lambda_{\{2,3\}} + \lambda_N = 1,$$

$$\lambda_{\{3\}} + \lambda_{\{1,3\}} + \lambda_{\{2,3\}} + \lambda_N = 1.$$

For the general case, we shall write the dual problem in the following way:

$$\text{maximize} \sum_{S \subset N} \lambda_S v(S), \tag{5.7}$$

$$\text{subject to } \lambda_S \geq 0 \quad \text{and} \tag{5.8}$$

$$\sum_{S: i \in S \subset N} \lambda_S = 1 \quad \text{for all } i = 1, \ldots, n. \tag{5.9}$$

Since the primal problem has solution, from the fundamental duality theorem in linear programming we know that that this problem too is feasible and has solutions; furthermore, there is no duality gap. We can then claim:

Theorem 5.2.2 *The core $C(v)$ of the game v is nonempty if and only if every vector $(\lambda_S)_{S \subset N}$ fulfilling the conditions:*

$$\lambda_S \geq 0 \ \forall S \subset N \quad \text{and}$$

$$\sum_{S: i \in S \subset N} \lambda_S = 1 \quad \text{for all } i = 1, \ldots, n,$$

verifies also:

$$\sum_{S \subset N} \lambda_S v(S) \leq v(N).$$

At a first reading the above result could look uninteresting: it is not clear why solving the dual problem should be easier than solving the initial one. However, as often in game theory, it has a very appealing interpretation, which can convince us to try going further in the analysis. First of all, let us observe that we can give an interpretation to the coefficients λ_S. The n conditions:

$$\sum_{S: i \in S \subset N} \lambda_S = 1 \quad \text{for all } i = 1, \ldots, n$$

(together with the nonnegativity constraints) suggest to look at these coefficient as how much (in"percentage") a coalition represents the players: $\lambda_{\{1,2\}}$ represents, for instance, the percentage of participation of players one and two to the coalition $\{1, 2\}$. Of course, every player must be fully represented in the game. Thus, in a sense, the theorem suggests that, no matter the players decide their quota in the coalitions, the corresponding weighted values must not exceed the available amount of utility $v(N)$. It is clearly a way to control the power of the intermediate coalitions.

The geometry of the set of λ_S fulfilling the above constraints is quite clear: we have to intersect various planes with the cone made by the first octant: as a result we get a convex polytope, which has a finite number of extreme points, the interesting points to look for when one must maximize a linear function. The very important fact is that the theory is able to characterize these points. We do not go in many details here, but rather we just describe the situation.

A family $(S_1, \dots, S_m)$ of coalitions (i.e. a subset of 2^N) is called *balanced* provided there exists $\lambda = (\lambda_1, \dots, \lambda_m)$ such that $\lambda_i > 0$ $\forall i = 1, \dots, m$ and, for all $i \in N$:

$$\sum_{k:i \in S_k} \lambda_k = 1.$$

λ is called a *balancing* vector (for the given balanced family).

Example 5.2.3 A *partition* of N (i.e. any family of disjoint sets covering N) is a balancing family, with balancing vector made by all 1. Let $N = \{1, 2, 3, 4\}$; the family $(\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4\})$ is balanced, with vector $(1/2, 1/2, 1/2, 1)$. Let $N = \{1, 2, 3\}$, and consider the family $(\{1\}, \{2\}, \{3\}, N)$. It is balanced, and every vector of the form $(p, p, p, 1 - p)$, $0 < p < 1$, is a balancing vector. The family $(\{1, 2\}, \{1, 3\}, \{3\})$ is not balanced. Observe that, given a vector $\lambda = (\lambda)_S$ fulfilling the constraint inequalities (5.8) and (5.9), the positive coefficients in it are the balancing vectors of a balanced family. For instance, always for the three player case, the vector $\{\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{3}{5}\}$ corresponds to the family $\mathcal{B} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{N\}$, the vector $\{0, 0, 0, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{3}{5}\}$ corresponds to the family $\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{N\}$.

Observe that in the case of the partition the balancing vector is unique, while it is not in the third example above. There is a precise reason for this. It is clear that in the third example we could erase some members of the collection (f.i. N) and still have a balanced family. It is not possible to erase, instead, a coalition from, as an example, a partition, without destroying balancedness. Thus we can distinguish between *minimal* and not minimal balancing families. The minimal ones are characterized by the fact that the balancing vector is unique. The following Lemma is crucial in characterizing the extreme points of the constraint set in (5.8) and (5.9).

Lemma 5.2.1 *The positive coefficient of the extreme points of the constraint set in (5.8) and (5.9) are the balancing vectors of the minimal balanced coalitions.*

Proof. (Outline). Let $(\lambda_S)_{S \subset N}$ be a vector fulfilling (5.8) and (5.9). Consider the positive coefficients in it and the corresponding balanced family $\mathcal{B}$. There are two possibilities, either $\mathcal{B}$ is minimal or it is not. If it is not minimal, it contains a minimal balanced family (this should be proved), say $\mathcal{C}$, with balancing vector say μ, which we shall complete to $(\mu_S)_{S \subset N}$, by adding zero's in the suitable entries. Now observe that $\mu_S > 0$ implies $\lambda_S > 0$, and consider the two following vectors: $u = (1-t)\lambda + t\mu$, $v = (1 + t)\lambda - t\mu$. It is immediate to see that both fulfill (5.9). But it is also easy to see that for small $t > 0$ they also fulfill (5.8), due to the fact that, as we observed, $\mu_S > 0$ implies $\lambda_S > 0$. But this shows that λ is not an extreme point, since it is the midpoint of the segment joining u and v. Conversely, suppose $\mathcal{B}$ is minimal and there are u v fulfilling (5.8) and (5.9) and such that λ is the mid point of the segment joining u and v. Observe that, due to the non negativity constraints, if $\lambda_S = 0$, then also $u_S = 0$ and $v_S = 0$. Consider then $\mathcal{C} = \{S : u_S > 0\}$ and $\bar{\mathcal{C}} = \{S : v_S > 0\}$. It is not difficult to see that they are balanced families contained in $\mathcal{B}$ and since this last one is minimal, it follows that $\mathcal{B} = \mathcal{C} = \bar{\mathcal{C}}$. Since they are minimal, the balancing coefficient are unique, and this implies $u = \lambda = v$, and this ends the proof. ∎

Thus the following theorem holds:

Theorem 5.2.3 *The cooperative game v has a nonempty core if and only if, for every minimal balanced collection of coalitions $(S_1, \dots, S_m)$, with balancing vector $\lambda = (\lambda_1, \dots, \lambda_m)$:*

$$\sum_{k=1}^{m} \lambda_k v(S_k) \leq v(N).$$

Now, the (absolutely non trivial) task is to see how many minimal balanced collections an N person game has. And also to observe, in order to facilitate our job, that partitions, which are minimal and balanced, can be ignored if we assume that the game is super additive, because in such a case the condition required in the theorem is automatically fulfilled. Let us fully develop the case of a three player game. Let us put:

$$\lambda_{\{1\}} = a,\ \lambda_{\{2\}} = b,\ \lambda_{\{3\}} = c,$$
$$\lambda_{\{1,2\}} = x,\ \lambda_{\{1,3\}} = y,\ \lambda_{\{2,3\}} = z,$$
$$\lambda_N = w.$$

The system of inequalities becomes:

$$a + x + y + w = 1,$$
$$b + x + z + w = 1,$$
$$c + y + z + w = 1.$$

Taking into account the non negativity conditions, we have the following extreme points (we conventionally assign zero to a coalition not involved in the balanced family):

$$(1, 1, 1, 0, 0, 0, 0) \quad \text{with balanced family} \quad (\{1\}, \{2\}, \{3\}),$$
$$(1, 0, 0, 0, 0, 1, 0) \quad \text{with balanced family} \quad (\{1\}, \{2, 3\}),$$
$$(0, 1, 0, 0, 1, 0, 0) \quad \text{with balanced family} \quad (\{2\}, \{1, 3\}),$$
$$(0, 0, 1, 1, 0, 0, 0) \quad \text{with balanced family} \quad (\{3\}, \{1, 2\}),$$
$$(0, 0, 0, 0, 0, 0, 1) \quad \text{with balanced family} \quad (N),$$
$$(0, 0, 0, (1/2), (1/2), (1/2), 0) \quad \text{with balanced family} \quad (\{1, 2\}, \{1, 3\}, \{2, 3\}).$$

Only the last one corresponds to a balanced family not being a partition of $N = \{1, 2, 3\}$. Thus, if the game is super additive, we have just one condition to check: the core is non empty provided:

$$v(\{1, 2\}) + v(\{1, 3\}) + v(\{2, 3\}) \leq 2v(N).$$

This is not difficult. The situation however becomes very quickly much more complicated if the game is played by a bigger number of players. For instance, in the case of 4 players, after some simplification, it can be shown that 11 inequalities must be checked to be true in order to have a non empty core.

5.3 The nucleolus

Another important solution concept for cooperative games is the so called nucleolus. Let us introduce it. For an imputation x and a coalition A, let us denote by $e(A, x)$ the excess of A over x, i.e.

$$e(A, x) = v(A) - \sum_{i \in A} x_i.$$

$e(A, x)$ is a measure of the dissatisfaction of the coalition A with respect to the assignment of x. It is clear that if $e(A, x) > 0$ the coalition complains about the distribution x of utilities, since alone it is able to do better. On the contrary, $e(A, x) < 0$ is a sign that the imputation x is interesting for the players of the coalition A. Now, define the following order for vectors in some fixed Euclidean space: $x \leq_L y$ provided either $x = y$ or there exists $j \geq 1$ such that $x_i = y_i$ for all $i < j$, and $x_j < y_j$ (a total order, called the *lexicographic* order).

Now, for an imputation x, denote by $\theta(x)$ the vector with $2^n - 1$ components, such that $\theta_1(x) \geq \theta_2(x) \geq \dots$ and $\theta_i(x) = e(A, x)$, for some $A \subset N$. In other words, $\theta(x)$ arranges in decreasing order the excesses of the coalitions over the imputation x.

We are ready to define the nucleolus.

Definition 5.3.1 *The* nucleolus ν *of the game* G *is the set of the imputations* x *such that* $\theta(x) \leq_L \theta(y)$*, for all* y *imputations of the game* v.

Let us see an example.

Example 5.3.1 Let us find the nucleolus of the simple, three players, majority game. Suppose $x = (a, b, 1 - a - b)$, with $a, b \geq 0$ and $a + b \leq 1$. It is clear that, not matter a, b are, the coalitions S complaining $(e(S, \emptyset) > 0)$ are those with two members. $e(\{1, 2\}) = 1 - (a + b)$, $e(\{1, 3\}) = 1 - (a + 1 - a - b)$, $e(\{2, 3\}) = 1 - (b + 1 - a - b)$. Thus we must minimize the greatest between the three quantities:

$$1 - a - b, \qquad b, \qquad a.$$

Suppose no one of them is $1/3$. Then at least one is greater than $1/3$. Then $\theta_1(x) > 1/3$ if $x \neq (1/3, 1/3, 1/3)$. It follows that $\nu = (1/3, 1/3, 1/3)$.

And now the main result concerning the nucleolus.

Theorem 5.3.1 *For any given* $v \in G$*, the nucleolus* $\nu(v)$ *is a singleton.*

Proof. We define $2^n - 1$ sets in the following way: let

$$I_1(v) = \{x \in I(v) : \theta_1(x) \leq \theta_1(y), \forall y \in I(v)\};$$

$$I_j(v) = \{x \in I_{j-1}(v) : \theta_j(x) \leq \theta_j(y), \forall y \in I_{j-1}(v)\};$$

for $j = 2, \dots, 2^n - 1$. The following facts are simple to check: the sets $I_j(v)$ are all nonempty. This follows from the fact that $I(v)$ is nonempty compact (convex) and that θ_j is a continuous function. Moreover, it is clear that $I_{2^n-1}(v)$ is the nucleolus: simply think to which vectors are in I_1: exactly those minimizing the maximum excess. If this set reduces to a singleton, it is the nucleolus, by definition (of lexicographic order). Otherwise, in I_2 there are those vectors from I_1 minimizing the second maximum excess. And so on. Writing a formal proof is more difficult than understanding, so I stop here. About uniqueness, of the vector in the nucleolus. It is clear that if two different vectors x, y are in $I_{2^n-1}(v)$, then $\theta(x) = \theta(y)$. Now, suppose the maximum excess for x, y is a and that k coalitions have a as maximum excess. Suppose also that $j < k$ coalitions are the same for x, y but the other are different. Write

$$\theta(x) = (e(A_1, x), \dots, e(A_{j-1}, x), e(A_j, x), \dots, e(A_k, x), e(A_{k+1}, x), \dots, e(A_{2^n-1}, x))$$

$$\theta(y) = (e(A_1, y), e(A_{j-1}, x), \dots, e(B_j, y), \dots, e(B_k, x), \dots, e(B_{2^n-1}, y)),$$

where $A_l \neq B_m$ for $j \leq l, m \leq k$. Now consider $e(S, \frac{x+y}{2})$. If $S = A_l$ for some $1 \leq l \leq j$, then

$$e(S, \frac{x+y}{2}) = v(A_l) - \sum_{i \in A_l} \frac{x_i + y_i}{2} = \frac{1}{2}[(v(A_l) - \sum_{i \in A_l} x_i) - (v(A_l) - \sum_{i \in A_l} y_i)] = a.$$

Otherwise observe that either $v(S) - \sum_{i \in S} x_i$ or $v(S) - \sum_{i \in S} y_i$ (or both) are strictly smaller than a. It follows $\theta(\frac{x+y}{2}) <_L \theta(x)$, since the $j-th$ component of $\theta(\frac{x+y}{2})$ is smaller than a, while it is a the same one for $\theta(x)$. But this is impossible. Now, we must consider the case when the two sets of coalitions giving a as maximum excess for x and y are the same. But one can repeat the same argument with the second maximum excess $b < a$. By repeating the argument, since we are supposing $x \neq y$, we must find one of these values for which a difference between the two sets of coalition occurs, and this contradiction ends the proof. ■

Now a property of the nucleolus which is much easier to prove, and equally nice.

Proposition 5.3.1 *Suppose v is such that $C(v) \neq \emptyset$. Then $\nu(v) \in C(v)$.*

Proof. Take $x \in C(v)$. Then $\theta_1(x) \leq 0$. Thus $\theta_1(\nu) \leq 0$. ■

5.4 Power indices

The third solution for games in G we present in these notes is the Shapley value. For sure, this is a very important solution concept, having a really wide range of applications. Also, the way it was defined become a standard approach to a solution concept in cooperative setting, and not only. Remember how Nash defined a bargaining solution: his idea was to find a short and reasonable pool of properties *characterizing* the solution, in the sense that the only solution fulfilling the list of properties is the proposed one. Why is this approach so important in cooperative game theory? As already mentioned, the solution concepts available in the literature are very many. Usually, they provide reasonable outcomes in some games, less intuitive results in other games. A way to try to understand the answer given by different solutions to the same game is to appeal to the pool of properties characterizing the solution itself. The Shapley value is important not only as a solution concept for general cooperative games. As already mentioned, a very important class of cooperative games is that one of the simple games. They are mainly intended to measure the strength of the players in situations where decisions must be taken, according to some majority rule: just to give an example, everybody knows that in a firm where there is one person having the 30% of the share and some other 100 people equally divide the rest, the

first shareholder has a position of almost total control on the firm. But it is important, for instance for legal questions, to give a more precise, quantitative idea of the power of the players in similar situations. Of course, in doing this there must be some arbitrary convention: nothing in this subject can be settled once for all. However, as frequently noticed when dealing with game theory, all this type of information, though less "objective" than in some other subjects, can be useful for shedding lights in difficult problems. So that, among the various solution concepts for cooperative games, the so called *power indices* are very important. The Shapley value is the first example of a power index; another important one is the Banzhaf value, which however does *not* satisfy, in general, the requirement to be an imputation, which makes it in general useless as a general solution concept for a game.
In the next sections we treat the Shapley value in more details, and we simply give a glance to the Banzhaf index.

5.4.1 The Shapley value

The Shapley value, introduced by Shapley in the early fifties, is a solution concept which has an important feature: it is a (single) vector for each game v. Let us see a possible way of defining it. We propose a set of properties characterizing it on the set $\mathcal{G}$.

Define a solution for the cooperative game any function $\phi : G \to \mathbb{R}^n$. Let us see a possible list of reasonable properties the function ϕ could fulfill.

1. For every $v \in \mathcal{G}$, $\sum_{i \in N} \phi_i(v) = v(N)$
2. Let v be a game with the following property, for players i, j: for every A not containing i, j, $v(A \cup \{i\}) = v(A \cup \{j\})$. Then $\phi_i(v) = \phi_j(v)$
3. Let $v \in \mathcal{G}$ and $i \in N$ be such that $v(A) = v(A \cup \{i\})$ for all A. Then $\phi_i(v) = 0$
4. for every $v, w \in \mathcal{G}$, $\phi(v + w) = \phi(v) + \phi(w)$.

The first axiom is simply *efficiency* and it is mandatory, as we already argued, talking about *solutions* in cooperative theory. The second one is *symmetry*: in the game v the two players i, j contribute to any coalition in exactly the same way, thus they must have the same utility (or power). The third one says that a *null* player, which is a player contributing nothing to any coalition, should have zero. More complicated is the interpretation of the fourth axiom, even if its mathematical meaning is quite simple and attracting: it says that ϕ acts additively (actually linearly, as we shall see) on the vector space $\mathcal{G}$.

And here is the Shapley theorem.

Theorem 5.4.1 *Consider the following function $\sigma : \mathcal{G} \to \mathbb{R}^n$:*

$$\sigma_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} \frac{s!(n-s-1)!}{n!} \left[v(S \cup \{i\}) - v(S) \right]. \tag{5.10}$$

Then σ is the only function ϕ fulfilling the above list of properties.

Proof. We divide the proof in two steps. The first one shows that there exists at most one function ϕ fulfilling the above properties. Then, we prove that σ does fulfill the properties.

First step The idea is simple. We shall prove that ϕ is univocally defined on a basis of the space $\mathcal{G}$, and then we exploit linearity of ϕ (a simple consequence of additivity and symmetry) to conclude. Consider a unanimity game u_A. How much assigns ϕ to the players? Suppose i, j are both in the coalition A. Then, applying the property 2. to any coalition T not containing i, j, we see that $v(T \cup \{i\}) = v(T \cup \{j\}) = 0$, and thus ϕ must assign the same amount to both. Moreover, a player not belonging to A is clearly a null one. Taking into account that $u_A(N) = 1$ must be equally divided among the players of the coalition A, we finally see that ϕ must assign $\frac{1}{|A|}$ to each player in A and nothing to the players not belonging to A. This means that ϕ is uniquely determined on the unanimity games. The same argument applies to the game cu_A, for $c \in \mathbb{R}$ and this, together with the additivity axiom, shows that there is at most one function fulfilling the properties.
Second step We now prove that σ fulfills the required properties, and this will end the proof.

- To prove 2. Suppose v is such that for every A not containing i, j, $v(A \cup \{i\}) = v(A \cup \{j\})$. We must then prove $\sigma_i(v) = \sigma_j(v)$. Write

$$\sigma_i(v) = \sum_{S \in 2^{N \setminus \{i \cup j\}}} \frac{s!(n-s-1)!}{n!} \left[v(S \cup \{i\}) - v(S)\right] +$$

$$+ \sum_{S \in 2^{N \setminus \{i \cup j\}}} \frac{s!(n-s-1)!}{n!} \left[v(S \cup \{i \cup j\}) - v(S \cup \{j\})\right],$$

$$\sigma_j(v) = \sum_{S \in 2^{N \setminus \{i \cup j\}}} \frac{s!(n-s-1)!}{n!} \left[v(S \cup \{j\}) - v(S)\right] +$$

$$+ \sum_{S \in 2^{N \setminus \{i \cup j\}}} \frac{s!(n-s-1)!}{n!} \left[v(S \cup \{i \cup j\}) - v(S \cup \{i\})\right].$$

  Now, the terms in the sums are equal, since by assumption for A not containing i, j, $v(A \cup \{i\}) = v(A \cup \{j\})$. Thus the Shapley index fulfills the symmetry axiom;
- The null player property is obvious;
- The linearity property is obvious;
- Let us prove efficiency (property 1.) Thus we need to consider the sum

$$\sum_{i=1}^{n} \sigma_i(v).$$

  Observe the following: the term $v(N)$ appears n times, one for each player, in the above sum, and it is always taken positive, with coefficient $\frac{(n-1)!(n-n)!}{n!} = \frac{1}{n}$. This means that in the sum the term $v(N)$ appears with coefficient one. Thus the proof will be complete if we show that, for $A \neq N$, the term $v(A)$ appears with positive and negative multiplicative coefficients which are the same. More precisely, it appears in the sum a times (once for every player in A), with coefficient

$$\frac{(a-1)!(n-a)!}{n!},$$

  providing

$$\frac{a!(n-a)!}{n!},$$

as positive coefficient for $v(A)$. Now consider when $v(A)$ appears with negative sign. This happens $n-a$ times (once for each player not in A) with coefficient

$$\frac{a!(n-a-1)!}{n!},$$

and the result is

$$\frac{a!(n-a)!}{n!}.$$

This concludes the proof. ■

The coefficient $\frac{(s)!(n-s-1)!}{n!}$ appearing in the formula of the Shapley value has a nice probabilistic interpretation. Suppose the players plan to meet at a certain place at a fixed hour, and suppose the order of arrival is equally likely. Suppose moreover the player i enters into the coalition S if and only if he finds all members of S and only them when arriving. Well, the coefficient represents exactly the probability that this fact happens.

In the case of the simple games, the Shapley value can be rewritten in the following form:

$$\sigma_i(v) = \sum_{A \in \mathcal{A}_i} \frac{(a-1)!(n-a)!}{n!},$$

where $\mathcal{A}_i$ is the set of the coalitions A such that:

- $i \in A$;
- A is winning;
- $A \setminus \{i\}$ is not winning.

Example 5.4.1 Let us compute, in different ways, the Shapley value of the following game:

$$v(\{1\}) = 0, v(\{2\}) = v(\{3\}) = 1, v(\{1,2\}) = 4, v(\{1,3\}) = 4, v(\{2,3\}) = 2, v(N) = 8.$$

Let us compute the Shapley value by writing v as a linear combination of the games u_A. Let us calculate the coefficients c_A. We have $c_{\{1\}} = 0$, $c_{\{2\}} = 1$, $c_{\{3\}} = 1$, $c_{\{1,2\}} = v_{\{1,2\}} - c_{\{1\}} - c_{\{2\}} = 3$, $c_{\{1,3\}} = 3$, $c_{\{2,3\}} = 0$, $c_{\{N\}} = 0$. Thus $S_1(v) = 0 + 0 + \frac{3}{2} + \frac{3}{2} = 3$, $S_2(v) = 1 + 0 + \frac{3}{2} + 0 = \frac{5}{2}$, $S_3(v) = 0 + 1 + 0 + \frac{3}{2} = \frac{5}{2}$. The second method we use is with the probabilistic interpretation:

	1	2	3
123	0	4	4
132	0	4	4
213	3	1	4
231	6	1	1
312	3	4	1
321	6	1	1
	$\frac{18}{6}$	$\frac{15}{6}$	$\frac{15}{6}$

.

Finally, let us use the formula (5.10). We get:

$$\sigma_1(v) = \frac{1!1!}{3!}[v(\{1,2\}) - v(\{2\})] + \frac{1}{6}[v(\{1,3\}) - v(\{3\})] + \frac{1}{3}[v(\{N\}) - v(\{2,3\})] = 3,$$

and analogously,

$$\sigma_2(v) = \frac{2}{6} + \frac{5}{6} + \frac{4}{3} = \frac{15}{6},$$
$$\sigma_3(v) = \frac{2}{6} + \frac{5}{6} + \frac{4}{3} = \frac{15}{6}.$$

Observe that the first player, which is weaker than the other ones when alone, has a stronger power index, and this reflects the fact that he is a strong player when he makes a coalition with another player. Finally, the fact that the second and the third one get the same index could have been forecast, since the two players have the same strength in the game, and thus by symmetry they must have the same power. This means that actually it was enough to evaluate S_1 in order to have S.

Example 5.4.2

$$v(\{1\}) = 0, v(\{2\}) = v(\{3\}) = 1, v(\{1,2\}) = 4, v(\{1,3\}) = 4, v(\{2,3\}) = 2, v(N) = 9.$$

By using the formula:

$$\sigma_1(v) = \frac{(3-1)!}{3!}(v(1) - v(\emptyset)) + \frac{1!(3-2)!}{3!}(v(1,2) - v(2)) + \frac{1!(3-2)!}{3!}(v(1,3) - v(3)) +$$
$$+ \frac{2!1!}{3!}(v(1,2,3) - v(2,3)) = \frac{20}{6};$$

Given the symmetry of players 2 and 3, and exploiting efficiency:

$$\sigma_2(v) = \sigma_3(v) = \frac{17}{6}.$$

5.4.2 The Banzhaf value

The Shapley value, in order to evaluate the power index of the player i, takes into account the marginal contribution $m_i(v,S) = v(S \cup \{i\}) - v(S)$ that player i provides to any coalition S, weighting them according to a probabilistic coefficient, depending from the size of the coalition itself. This can be easily generalized. A power index ψ is called a *probabilistic index* provided for each player i there exists a probability measure p_i on $2^{N \setminus \{i\}}$ such that

$$\psi_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} p_i(S) m_i(v,S). \tag{5.11}$$

In the set of the probabilistic indices, there is an important subfamily, that one of the *semivalues*, i.e those indices such that the coefficient $p_i(S)$ does not depend from the player i, and depend only from the *size* of the coalition S: $p_i(S) = p(s)$. Furthermore, if $p(s) > 0$ for all s, then the semivalue is called *regular semivalue*. It is not difficult to see that the Shapley index is a regular semivalue, with

$$p(s) = \frac{1}{n\binom{n-1}{s}}.$$

The Shapley value σ has the noteworthy property of being the only regular semivalue which is also an imputation, i.e. it fulfills the condition $\sum_i \sigma_i(v) = v(N)$. Another important regular semivalue is the *Banzhaf index* β, defined as

$$\beta_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} \frac{1}{2^{n-1}}(v(S) - v(S \setminus \{i\})). \tag{5.12}$$

In this case the coefficient $p(s)$ is constant: it is assumed that the player i has the same probability to join any coalition. Clearly, it is possible to provide a pool of properties characterizing this value: here we only observe that there is only one property of the above list characterizing the Shapley value which is *not* satisfied by Banzhaf's: efficiency. Now, let us see an example:

Example 5.4.3 We evaluate the Banzhaf index of two examples considered before in case of Shapley, in order to compare them. First game:

$$v(\{1\}) = 0, v(\{2\}) = v(\{3\}) = 1, v(\{1,2\}) = 4, v(\{1,3\}) = 4, v(\{2,3\}) = 2, v(N) = 8.$$

Using the formula (5.12):

$$\beta_1 = \frac{3}{4} + \frac{6}{4} + \frac{3}{4} = 3$$
$$\beta_2 = \frac{1}{4} + 1 + \frac{1}{4} + 1 = \frac{5}{2}$$
$$\beta_3 = \frac{1}{4} + 1 + \frac{1}{4} + 1 = \frac{5}{2}$$

We see that it coincides with the Shapley index. Let us consider now:

$$v(\{1\}) = 0, v(\{2\}) = v(\{3\}) = 1, v(\{1,2\}) = 4, v(\{1,3\}) = 4, v(\{2,3\}) = 2, v(N) = 9.$$

Then:

$$\beta_1 = \frac{3}{4} + \frac{7}{4} + \frac{3}{4} = \frac{13}{4}$$
$$\beta_2 = \frac{1}{4} + 1 + \frac{1}{4} + \frac{5}{4} = \frac{11}{4}$$
$$\beta_3 = \frac{1}{4} + 1 + \frac{1}{4} + \frac{5}{4} = \frac{11}{4}$$

In this case the Banzhaf index provides a different answer with respect to Shapley's.

Example 5.4.4 The ONU security council Let $N = \{1, \ldots, 15\}$ be the set of the members of the council. The permanent members $1, \ldots 5$ are veto players, and a motion is accepted provided it gets 9 votes, including the five votes of the permanent members. Let i be a player which is no veto. His marginal value is 1 if and only if it enters a coalition A such that

1. A contains the 5 veto players
2. $a = 8$.

The number of such coalitions is the number of a possible choice of three elements (the three accompanying the five veto players) out of nine (players which are no veto and no i). Thus

$$\sigma_i = \frac{8! \cdot 6! \cdot 9 \cdot 8 \cdot 7}{15! \cdot 3 \cdot 2} \simeq 0.0018648.$$

The power of the veto players can be calculated by difference and symmetry. The result is $\sigma_i \simeq 0,1962704$. The ratio between the two is $\simeq 105.25$. Thus the Shapley value asserts that the power of the veto players is more than one hundred times the power of the other players. Calculating Banzhaf's, we get that

$$\beta_i = \frac{1}{2^{14}} \binom{9}{3} = \frac{21}{2^{12}} \simeq 0.005127,$$

for a non-veto player, while if i is a veto player we get:

$$\beta_i = \frac{1}{2^{14}} \left(\binom{10}{4} + \dots \binom{10}{10} \right) = \frac{1}{2^{14}} \left(2^{10} - \sum_{k=0}^{3} \binom{10}{k} \right) = \frac{53}{2^{10}} \simeq 0.0517578,$$

and the ratio is $\simeq 10.0951$. The difference between the two indices is quite important.

The Shapley and Banzhaf indices behave differently on unanimity games. As we have seen, Shapley assigns $1/s$ to every player in S, in the game u_S, while Banzhaf assigns $\frac{1}{2^{s-1}}$ to the same players. A family of regular semivalues can be defined by assigning $\frac{1}{s^a}$, $a > 0$, to the non null players in the unanimity game u_S. Another interesting family of regular semivalues is defined by setting

$$p_\alpha(s) = \alpha^{s-1}(1 - \alpha^{n-s}),$$

where $\alpha \in (0, 1)$.

5.5 Exercises

Exercise 5.5.1 Some useful definitions:

A game $G = (N, v)$ is *monotonic* if $v(S) \leq v(T)$, $\forall S \subseteq T$.

A game $G = (N, v)$ is *convex* if it holds

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T), \ \forall S, T \subseteq N$$

A game $G = (N, v)$ is *simple* provided v is valued on $\{0, 1\}$, it is monotonic and $v(N) = 1$. A coalition with value 1 is said *winning*, *losing* otherwise.

A game $G = (N, v)$ is *coesive* if for each partition of N $\{S_1, \dots, S_k\}$, then

$$\sum_{i=1,\dots,k} v(S_i) \leq v(N).$$

A game $G = (N, v)$ is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$, $\forall S, T \subseteq N$, s.t. $S \cap T = \emptyset$.

Prove the following:

1. a convex game is superadditive
2. a superadditive game is coesive (in particular a convex game is coesive).

Solution

1. From the definition, a convex game (N, v) has the property
$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \quad \forall S, T \subseteq N$$
in particular, when $S \cap T = \emptyset$,
$$v(S) + v(T) \leq v(S \cup T)$$
i.e. the game is superadditive
2. consider a partition of N $\{S_1, \ldots, S_k\}$, $S_i \cap S_j = \emptyset$ for each $i \neq j$ and $S_1 \cup \ldots \cup S_n = N$. The game is superadditive, then it holds
$$v(S_1) + v(S_2) \leq v(S_1 \cup S_2)$$
but then
$$v(S_1 \cup S_2) + v(S_3) \leq v(S_1 \cup S_2 \cup S_3)$$
iterating the process, we get
$$\sum_{i=1,\ldots,k} v(S_i) \leq v(N).$$

Exercise 5.5.2 Say which properties the following weighted majority game satisfies: $[51; 12, 34, 40, 14]$.

Solution We have a weighted majority game, with $N = \{1, 2, 3, 4\}$. The winning coalitions are $\{1, 3\}$, $\{2, 3\}$, $\{3, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$ and $\{1, 2, 3, 4\}$, with value 1, all the others are loosing, with value 0.

- the game is monotonic;
- the game is simple;
- the game is not convex (take $S = \{1, 3\}$ and $T = \{3, 4\}$);
- the game is superadditive $\Rightarrow$ coesive.

Exercise 5.5.3 Bankruptcy Game A bankruptcy game is defined by the triple $B = (N, c, E)$, where $N = \{1, \ldots, n\}$ is the set of creditors, $c = \{c_1, \ldots, c_n\}$ is the credits claimed by them and E is the available capital. The bankruptcy condition is then $E < \sum_{i \in N} c_i = C$.
We have two different ways to define a TU game for this situation: one pessimistic and one optimistic.
N is the set of players for both, while the two characteristic functions are given by
$$v_P(S) = \max\left(0, E - \sum_{i \in N \setminus S} c_i\right) \quad S \subseteq N$$
for the pessimistic game and
$$v_O(S) = \min\left(E, \sum_{i \in S} c_i\right) \quad S \subseteq N$$
for the optimistic one.

1. Write the characteristic functions for the pessimistic game and for the optimistic one in the following situation: $(\{1,2\},(3,4),5)$. Which one is better?
2. Show that the core of the pessimistic game coincides with the admissibility conditions.

Solution

1.
$$v_O(1) = 3,\ v_O(2) = 4,\ v_O(12) = 5;$$
$$v_P(1) = 1,\ v_P(2) = 2,\ v_O(12) = 5;$$
The optimistic game says that the two players, separately, can obtain $3+4=7$, while the total amount is just 5. The optimistic game is not realistic
2. We have to show that
$$x \in C(v_P) \Leftrightarrow \begin{cases} \sum_{i\in N} x_i = E \\ 0 \le x_i \le c_i \ i \in N \end{cases}$$
$\Rightarrow$ The first condition is the efficiency. For the second one $\forall i \in N$ we get $x_i \ge v_P(i) \ge 0$ and $E - x_i = \sum_{j\in N\setminus\{i\}} x_j \ge v_P(N\setminus\{i\}) \ge E - c_i \Rightarrow x_i \le c_i$.
$\Leftarrow$ The condition of efficiency is obviously satisfied. $\forall S \subseteq N$ we have
- if $v_P(S) = 0 \le \sum_{i\in S} x_i$
- if $v_P(S) = E - \sum_{i\in N\setminus S} c_i \le E - \sum_{i\in N\setminus S} x_i = \sum_{i\in S} x_i$.

Exercise 5.5.4 Given the game (N,v) with $N=\{1,2,3\}$ and $v(\{i\})=0$, $v(\{1,2\}) = v(\{1,3\}) = 1$, $v(\{2,3\}) = 0$, $v(N)=2$, represent the core of the game.

Solution
$C(v) =$ convex hull of the set of vectors $\{(1,0,1),(0,1,1),(2,0,0),(1,1,0)\}$.[2]

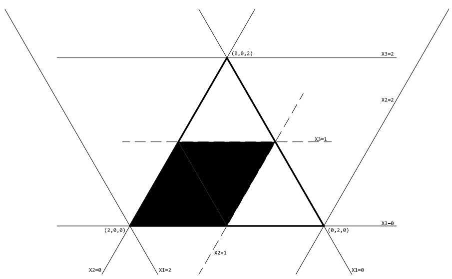


Fig. 5.1. Exercise 5.5.4

Exercise 5.5.5 Given a convex TU game (N,v), define the payoff profile x by $x_i = v(S_i \cup \{i\}) - v(S_i)$ for each $i \in N$, where $S_i = \{1,\dots,i-1\}$ (with $S_1 = \emptyset$). Show that $x \in C(v)$.

[2] The convex hull of a set of vectors is the smallest convex set containing the vectors. In this case it is a quadrilateral.

Solution Let $S^* = \{i_1, \ldots, i_{|S^*|}\}$ be any coalition with $i_1 < \ldots < i_{|S^*|}$. Then $x_{i_1} = v(S_{i_1} \cup \{i_1\}) - v(S_{i_1}) \geq v(S_{i_1})$ (take $S = S_{i_1}$ and $T = \{i_1\}$ in the definition of convexity). But then $x_{i_1} + x_{i_2} \geq v(\{i_1\}) + v(S_{i_2} \cup \{i_2\}) - v(S_{i_2}) \geq v(\{i_1, i_2\})$ (take $S = S_{i_2}$ and $T = \{i_1, i_2\}$ in the definition of convexity). Continuing similarly we reach the conclusion that $x_{i_1} + \ldots + x_{|S^*|} \geq v(S^*)$. Moreover, $\sum_{i \in N} x_i = v(N)$, so that $x \in C(v)$. Observe, the same proof shows that any vector x constructed in the same way for any permutation of the players belong to the core.

Exercise 5.5.6 Show that in a convex game the Shapley value is a member of the core.

Solution This follows from the result of the previous exercise, since the Shapley value is a convex combination of the vectors x like before.

Exercise 5.5.7 Given the TU game (N, v) with $N = \{1, 2, 3\}$ and $v(\{1\}) = 1$, $v(\{2\}) = v(\{3\}) = 2$, $v(\{1, 2\}) = 5$, $v(\{1, 3\}) = 4$, $v(\{2, 3\}) = 7$, $v(N) = 8$:

1. find the core of the game
2. find the Shapley value;
3. find the Banzhaf value.

Solution

1. The core of the game is given by $C(v) = \{(1, 4, 3)\}$
2. The Shapley value is given by

$$\sigma_1 = \frac{1}{3}[1-0] + \frac{1}{6}[5-2] + \frac{1}{6}[4-2] + \frac{1}{3}[8-7] = \frac{3}{2}$$

$$\sigma_2 = \frac{1}{3}[2-0] + \frac{1}{6}[5-1] + \frac{1}{6}[7-2] + \frac{1}{3}[8-4] = \frac{7}{2}$$

$$\sigma_3 = \frac{1}{3}[2-0] + \frac{1}{6}[4-1] + \frac{1}{6}[7-2] + \frac{1}{3}[8-5] = \frac{6}{2}$$

3. The Banzhaf value is given by

$$\beta_1 = \frac{1}{4}[1-0] + \frac{1}{4}[5-2] + \frac{1}{4}[4-2] + \frac{1}{4}[8-7] = \frac{7}{4}$$

$$\beta_2 = \frac{1}{4}[2-0] + \frac{1}{4}[5-1] + \frac{1}{4}[7-2] + \frac{1}{4}[8-4] = \frac{15}{4}$$

$$\beta_3 = \frac{1}{4}[2-0] + \frac{1}{4}[4-1] + \frac{1}{4}[7-2] + \frac{1}{4}[8-5] = \frac{13}{4}$$

Exercise 5.5.8 Given the game with three players: $v(\{i\}) = 0$, $v(\{1, 2\}) = v(\{1, 3\}) = a$, $v(\{2, 3\}) = 2$, $v(N) = 3$, for which $a > 0$ is the core empty?

Solution When $a \leq 3$, the game is superadditive and the necessary and sufficient condition to have empty core is $v(\{1, 2\}) + v(\{1, 3\}) + v(\{2, 3\}) > 2v(N)$, providing $a > 2$, then $a < 2 \leq 3$. When $a > 3$ the game is non-monotonic and the core is empty. Then the core is empty when $a > 2$.

Exercise 5.5.9 Given the TU game (N, v) with $N = \{1, 2, 3\}$ and $v(\{i\}) = 0$, $v(\{1, 2\}) = v(\{1, 3\}) = 1$, $v(\{2, 3\}) = 0$, $v(N) = a$, with $a \geq 1$:

1. find the Shapley value
2. find the Banzhaf value
3. find the nucleolus.

Solution

1. We notice that players 2 and 3 are symmetric, then $\sigma_2 = \sigma_3 = \frac{1}{6} + \frac{1}{3}(a-1) = \frac{1}{3}a - \frac{1}{6}$. As the Shapley value is an imputation, $\sigma_1 = a - \sigma_2 - \sigma_3$, then $\sigma = \left(\frac{1}{3}a + \frac{2}{6}, \frac{1}{3}a - \frac{1}{6}, \frac{1}{3}a - \frac{1}{6}\right)$
2. $\beta = \left(\frac{1}{4}a + \frac{1}{2}, \frac{1}{4}a . \frac{1}{4}a\right)$
3. as players 2 and 3 are symmetric, the nucleolus is an imputation of the form $(a - 2x, x, x)$, with $0 \leq x \leq \frac{a}{2}$. The relevant excesses are $2x - a$, $-x$, $1 - a + x$. The straight lines $2x - a$ and $-x$ cross in $\left(\frac{a}{3}, -\frac{a}{3}\right)$. If $a > 3$, $\nu(v) = \left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$, otherwise $\nu(v) = \left(1, \frac{a-1}{2}, \frac{a-1}{2}\right)$.

Exercise 5.5.10 Given the TU game (N, v) with $v(S) = s^2$ where $s = |S|$, for each $S \subseteq N$, $S \neq \emptyset$:

1. Prove that the Shapley value is always equal to n for each player and for each n
2. is the core empty when $n = 3$?

Solution

1. For each n, $v(N) = n^2$ and, due to the symmetry of the players, since $\sum \sigma_i = n^2$, $\sigma = (n, \dots, n)$
2. when $n = 3$, since the game is superadditive, the core is nonempty if and only if the following inequality holds:

$$v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) \leq 2v(N).$$

Thus, since

$$4 + 4 + 4 \leq 18,$$

then the core is nonempty.

Exercise 5.5.11 Given the TU game (N, v) with $N = \{1, 2, 3, 4\}$ and

$$v(A) = \begin{cases} 1 \; |A| \geq 3, \; \{1,2\} \subset A \\ 0 \text{ otherwise} \end{cases}$$

1. say how many non empty coalitions there are s.t. $v(A) = 0$
2. find the Shapley value
3. find the core
4. find the nucleolus.

Solution This is a simple game, with 1 and 2 as veto players. Moreover also 3 and 4 are symmetric players.

1. The coalitions with value 1 are: $\{1,2,3\}$, $\{1,2,4\}$ and $\{1,2,3,4\}$. Then the non empty coalitions with value zero are $2^4 - 3 - 1 = 12$
2. it is enough to evaluate the index for one player, for example player 1. $\sigma_1 = \frac{1}{12} + \frac{1}{12} + \frac{1}{4} = \frac{5}{12}$. Then the Shapley value is $\sigma = \left(\frac{5}{12}, \frac{5}{12}, \frac{1}{12}, \frac{1}{12}\right)$
3. since 1 and 2 are veto players the core is given by $C(v) = co\{(1,0,0,0), (0,1,0,0)\}$

4. the nucleolus lies in the core, moreover the two veto players are symmetric, and thus $\nu(v) = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$.

Exercise 5.5.12 Given the TU game with n players: $v(\{i\}) = 0$, $v(S) = |S| - 1$ $\forall i = 1, \dots, n$ and $\forall S$ s.t. $|S| \geq 2$:

1. find the Shapley value
2. find the nucleolus
3. find the Banzhaf value.

Solution

1. Because of the symmetry and the fact that the Shapley value and the nucleolus are imputations, they both are of the form $(\frac{n-1}{n}, \dots, \frac{n-1}{n})$
2. because of the symmetry each player gets the same amounts: for all i

$$\beta_i = \frac{1}{2^{n-1}} \sum_{S \subseteq N, i \in S, S \neq \{i\}} 1 = \frac{2^{n-1} - 1}{2^{n-1}} = 1 - \frac{1}{2^{n-1}}$$

Exercise 5.5.13 Write a suitable characteristic function for the following cooperative games:

1. Four people own gloves, the first one has one left glove, the second one two left gloves, the third and fourth one have a right glove each
2. m people have one right glove, n people a left glove, with $n > m$
3. the airport of Erehwon needs a new landing field, since three companies decided to join the city. The first one needs 1 Km long landing field, whose cost is c_1, the second one 1,5 km, whose cost is c_2, the third one 2 km, whose cost is c_3.

Exercise 5.5.14 Find the core of the following games, drawing also a picture:

$$v(\{1\}) = 0, v(\{2\}) = -1, v(\{3\}) = 1, v(\{1,2\}) = 3, v(\{1,3\}) = 2, v(\{2,3\}) = 4, v(\{N\}) = 5.$$

Exercise 5.5.15 Given the cooperative game with n players, $n > 2$, defined as:

$$v(A) = \begin{cases} 1 & \text{if} \quad A \supset \{i, n\} \quad i \neq n \\ 1 & \text{if} \quad A = \{1, 2, \dots, n-1\} \\ 0 & \text{otherwise} \end{cases}$$

1. find the number of coalitions A such that $v(A) = 1$
2. find the core of the game.

Exercise 5.5.16 Find the Shapley and Banzhaf values for the two buyers one seller and one buyer two sellers games (see Example 5.1.2).

Exercise 5.5.17 Find the Shapley and Banzhaf values of the following games v_a, with $a > 0$:

$$v_a(\{i\}) = 0, \quad v_a(\{1,2\}) = v_a(\{2,3\}) = a, \quad v_a(\{1,3\}) = 1, \quad v_a(\{1,2,3\}) = 3.$$

Exercise 5.5.18 Find the Shapley value in the four player game where the first has a right glove, the second one has two right gloves, the third and fourth one have a left glove each.

Exercise 5.5.19 Find the Shapley and Banzhaf values for the weighted majority game $[6; 4, 3, 3]$.

Exercise 5.5.20 Consider the bankruptcy problem: A company goes bankruptcy owing 10 to the first creditor, 20 to the second one, 30 to the third one. It has however only 36. Find the core, the nucleolus and the Shapley and Banzhaf values.

6

More on zero sum games

In order to get the fundamental result concerning the existence of equilibrium, in mixed strategies, for the finite, zero sum games with two players, we need some results on convexity, that we briefly present in the following section.

6.1 Some useful facts in convexity

Definition 6.1.1 *A set $C \subset \mathbb{R}^n$ is said to be convex provided $x, y \in C$, $\lambda \in [0,1]$ imply:*

$$\lambda x + (1-\lambda)y \in C.$$

Convex sets enjoy very many useful properties: one of them is that a closed convex set with nonempty interior coincides with the closure of its internal points. This is no longer true for arbitrary sets. The two following pictures explain everything.

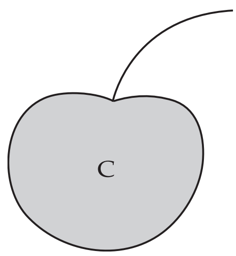


Fig. 6.1. cl int $C \neq$ cl C

The apple is a set C not fulfilling the property $C =$ cl int C. In the following picture, instead, we see that a segment joining an internal point of a convex set to any other point is contained in the interior of the set. As a consequence, we obtain that a closed convex set C with nonempty interior enjoys the property that $C=$ cl int C.[1]

[1] More generally, if C is convex with nonempty interior, it holds cl $C =$ cl int C.

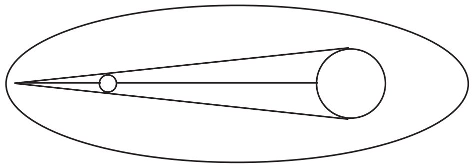


Fig. 6.2. Proving that $C = \text{cl int } C$ for convex sets with nonempty interior

Definition 6.1.2 *We shall call a* convex combination *of elements* $x_1, \dots, x_n$ *any vector* x *of the form*

$$x = \lambda_1 x_1 + \dots + \lambda_n x_n,$$

with $\lambda_1 \geq 0, \dots, \lambda_n \geq 0$ *and* $\sum_{i=1}^n \lambda_i = 1$.

We now see that a set C is convex if (and only if) it contains any convex combination of elements belonging to it.

Proposition 6.1.1 *A set* C *is convex if and only if for every* $\lambda_1 \geq 0, \dots, \lambda_n \geq 0$ *such that* $\sum_{i=1}^n \lambda_i = 1$, *for every* $c_1, \dots, c_n \in C$, *for all* n, *then* $\sum_{i=1}^n \lambda_i c_i \in C$.

Proof. Let

$$A = \{\sum_{i=1}^n \lambda_i c_i : \lambda_i \geq 0, \sum_i \lambda_i = 1, c_i \in C \, \forall i, n \in \mathbb{N}\}.$$

We must prove that $A = C$ if and only if C is convex. Observe that A contains C. Next, A is convex: just write the definition of convexity and verify. The proof will be concluded once we show that $A \subset C$ provided C is convex. To this end, take an element $x \in A$. Then

$$x = \sum_{i=1}^n \lambda_i c_i,$$

with $\lambda_i \geq 0, \sum_i \lambda_i = 1, c_i \in C$. If $n = 2$, then $x \in C$ just by definition of convexity. Suppose now $n > 2$ and that the statement is true for any convex combination of (at most) $n - 1$ element. Then

$$x = \lambda_1 c_1 + \dots + \lambda_n c_n = \lambda_1 c_1 + (1 - \lambda_1) y,$$

where

$$y = \frac{\lambda_2}{1 - \lambda_1} c_2 + \dots + \frac{\lambda_n}{1 - \lambda_1} c_n.$$

Now observe that y is a convex combination of $n - 1$ elements of C and thus, by inductive assumption, it belongs to C. Then $x \in C$ as it is a convex combination of two elements. ■

If C is not convex, then there is a smallest convex set containing C: it is the intersection of all convex sets containing C.

Definition 6.1.3 *The* convex hull *of a set* C, *denoted by* $\text{co}\, C$, *is defined as:*

$$\text{co}\, C \stackrel{\text{def}}{=} \bigcap_{A \in \mathcal{C}} A,$$

where $\mathcal{C} = \{A : C \subset A \wedge A \text{ is convex}\}$.

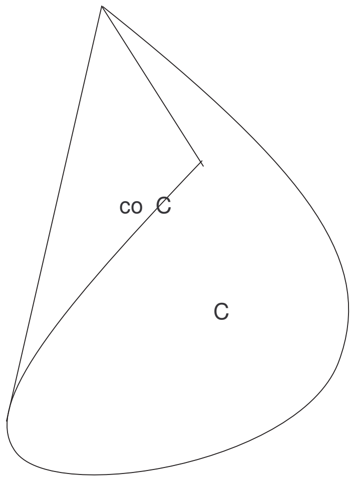


Fig. 6.3. The convex hull of C

Proposition 6.1.2 *Given a set C, then*

$$\text{co}\, C = \{\sum_{i=1}^{n} \lambda_i c_i : \ \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, c_i \in C\, \forall i, n \in \mathbb{N}\}.$$

The following result serves as an intermediate step to prove a very important theorem on convexity. We establish it here in a very particular case, and with an *ad hoc* proof, which is much simpler than that one in the general case, which is called the *Hahn-Banach separation theorem.*

Theorem 6.1.1 *Let C be a convex proper subset of the Euclidean space $\mathbb{R}^l$, let $\bar{x} \in$ cl C^c.*[2] Then there is an element $0 \neq x^* \in \mathbb{R}^l$ such that:

$$\langle x^*, c\rangle \geq \langle x^*, \bar{x}\rangle,$$

$\forall c \in C$

Proof. At first, suppose $\bar{x} \notin \text{cl}\, C$. The we can project $\bar{x}$ on $\text{cl}\, C$. Call p its projection.[3] Then

$$\langle p - \bar{x}, c - p\rangle \geq 0,$$

$\forall c \in C$, as it is geometrically easy to see (for a proof see Exercise 6.6.6). Setting $x^* = p - \bar{x}$, the above inequality can be written:

[2] Given a set $A \subset X$, we denote by A^c its complement, i.e. $A^c := \{x \in X : x \notin C\}$.

[3] i.e. p fulfills

1. $p \in C$
2. $\|\bar{x} - p\| \leq \|\bar{x} - c\|$ for all $c \in C$.

Such an element exists and is unique for every closed convex set C, for every $\bar{x} \notin C$.

$$\langle x^*, c - \bar{x} \rangle \geq \|x^*\|^2,$$

implying

$$\langle x^*, c \rangle \geq \langle x^*, \bar{x} \rangle$$

$\forall c \in C$, and this shows the claim in the particular case $\bar{x} \notin \overline{C}$. Observe, moreover, that we can choose $\|x^*\| = 1$. Now, if $\bar{x} \in \overline{C} \setminus C$, take a sequence $\{x_n\} \subset C^c$ such that $x_n \to \bar{x}$. From the first step of the proof, find norm one x_n^* such that

$$\langle x_n^*, c \rangle \geq \langle x_n^*, x_n \rangle,$$

$\forall c \in C$. Thus, possibly passing to a subsequence, we can suppose $x_n^* \to x^*$, where $\|x^*\| = 1$ (so that $x^* \neq 0$). Now take the limit in the above inequality, to get:

$$\langle x^*, c \rangle \geq \langle x^*, \bar{x} \rangle,$$

$\forall c \in C$. ■

Remark 6.1.1 A close look at the above proof shows that, in the case $\bar{x} \notin \overline{C}$, then actually it has been proved that there are an element $0 \neq x^* \in \mathbb{R}^l$ and $k \in \mathbb{R}$ such that:

$$\langle x^*, c \rangle > k > \langle x^*, \bar{x} \rangle,$$

$\forall c \in C$. Observe that this has a simple and important geometrical interpretation. Consider the set

$$\{x \in \mathbb{R}^l : \langle x^*, x \rangle = k\}.$$

Then the theorem asserts that this set, which is called *hyperplane* (actually a line if $l = 2$, a plane if $l = 3$), keeps on one side the point $\bar{x}$, on the other side the set C. This is called a *strict separation* of $\bar{x}$ and C. In the case $\bar{x} \in \overline{C} \setminus \operatorname{int} C$, the separation is instead obtained in a weak sense.

We state as a Corollary a part of the claim of the above Proposition, since it is a very important result.

Corollary 6.1.1 *Let C be a closed convex set in a Euclidean space, let x be on the boundary of C. Then there is a hyperplane containing x and leaving all of C in one of the halfspaces determined by the hyperplane.*

The hyperplane whose existence is established in Corollary 6.1.1 is said the be an hyperplane *supporting C at x*.

The following theorem extends the previous result, since it provides separation of two convex sets.

Theorem 6.1.2 *Let A, C be closed convex subsets of $\mathbb{R}^l$ such that* int A *is nonempty and* int $A \cap C = \emptyset$. *Then there is $0 \neq x^*$ such that*

$$\langle x^*, a \rangle \geq \langle x^*, c \rangle,$$

$\forall a \in A$, $\forall c \in C$.

Proof. Since $0 \in (\text{ int } A - C)^c$, we can apply Theorem 6.1.1 to find $x^* \neq 0$ such that

$$\langle x^*, x \rangle \geq 0,$$

$\forall x \in \text{int } A - C$. This amounts to saying that:

$$\langle x^*, a \rangle \geq \langle x^*, c \rangle,$$

$\forall a \in \text{int } A = A$, $\forall c \in C$. This implies

$$\langle x^*, a \rangle \geq \langle x^*, c \rangle,$$

$\forall a \in \text{cl int } A = A$, $\forall c \in C$, and this concludes the proof ■.

Definition 6.1.4 *Given a convex set C, a point $x \in C$ is said to be* extremal *if $x = \frac{u+v}{2}$, with $u, v \in C$ implies $u = v = x$.*

In other words, a point x in a convex set C is extremal if it is not the mid point of a non degenerate segment lying in C.

Theorem 6.1.3 *Let C be a closed convex and bounded subset of some Euclidean space. Then C is the convex hull of its extremal points.*

Proof. The proof is by induction on the dimension of the space where the convex set C lies. If this dimension is $n = 1$, then the claim is clear. Now, suppose the statement is true for closed convex sets lying in an $n-1$-dimensional space. Take any boundary point x of C. By the Corollary 6.1.1, there is a hyperplane H supporting C at x. Consider the set $C \cap H$: it is closed, convex, and belongs to an $n-1$-dimensional space. Thus x can be written as a convex combination of extremal points of C. The theorem then holds true for all boundary points of C. Now, consider any point c of C, take any extremal point x in the boundary of C and consider the line joining c and x. Since C is bounded, this line meets the boundary of C at another point, say y, which is convex combination of extremal points of C. As a result, being c a convex combination of x and y, which are convex combinations of extremal points of C, is itself a convex combination of extremal points of C. ■

The above proof shows something more; any point of a closed bounded convex set $C \subset \mathbb{R}^n$ can be written as a convex combination of *at most* $n+1$ extremal points of C.

The following result, known as the Farkas lemma, is very useful, and it is an easy consequence of the previous result.

Theorem 6.1.4 *Suppose there are $p+1$ vectors $v_1, \dots, v_{p+1}$ in $\mathbb{R}^n$ such that, for $z \in \mathbb{R}^n$:*

$$\langle z, v^k \rangle \geq 0 \quad \text{for } 1 \leq k \leq p \quad \Longrightarrow \quad \langle z, v^{p+1} \rangle \geq 0.$$

Then v^{p+1} lies in the closed convex cone C generated by $v^1, \dots, v^p$: there are $\alpha_1 \geq 0, \dots, \alpha_p \geq 0$ such that:

$$v^{p+1} = \sum_{j=1}^{p} \alpha_j v^j.$$

Proof. Suppose not. Then, it is possible to separate C from v^{p+1}. Thus there are $0 \neq z \in \mathbb{R}^n$ and c such that

$$\langle z, x\rangle \geq c > \langle z, v^{p+1}\rangle \ \ \forall x \in C.$$

Since the cone contains 0, it is $c \leq 0$. Moreover, suppose there is $x \in C$ such that $\langle z, x\rangle < 0$. Then, since $\lambda x \in C$ for all $\lambda > 0$, we would have $\lambda\langle z, x\rangle \to -\infty$, a contradiction showing that it is possible to take $c = 0$. ■

Observe that in the above proof we took for granted that the cone generated by a finite number of vectors is automatically closed; this is true, but not easy to prove, thus we omit its proof.

6.2 A proof of von Neumann theorem

In this section we prove the theorem of von Neumann (see Theorem 3.1.1) on the existence of equilibria for finite zero sum games. First of all, we remind its claim.

Theorem 6.2.1 *A two player, finite, zero sum game as described before has always equilibrium in mixed strategies.*

Proof. Suppose the game is described by an $n \times m$ matrix P, and denote by S_m, S_n the m^{th} and the n^{th}-simplexes respectively. Suppose also, w.l.o.g., that all the entries p_{ij} of the matrix P are positive. Now, consider the vectors $p_1, \dots, p_m$ of $\mathbb{R}^n$, where p_j denotes the j^{th} column of the matrix P. These vectors lie in the positive cone of $\mathbb{R}^n$. Call C the convex hull of these vectors, and set

$$Q_t \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n : x_i \leq t\},$$

for $i = 1, \dots, n$.

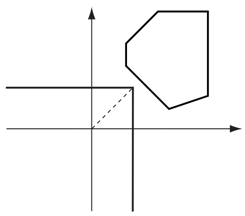


Fig. 6.4. The sets C and Q_t.

Observe that every element $\lambda \in S_m$ naturally provides an element of the set C, since C is the convex hull of the columns of the matrix. For, if $S_m \ni \lambda = (\lambda_1, \dots, \lambda_m)$, then $w = \sum \lambda_j p_j \in C$. Conversely, if $w \in C$, then there is (at least) an element $\lambda = (\lambda_1, \dots, \lambda_m) \in S_m$ such that $w = \sum \lambda_j p_j$.
Now set

$$v = \sup\{t \geq 0 : Q_t \cap C = \emptyset\}.$$

It is easy to see that Q_v and C can be (weakly) separated by an hyperplane: there are coefficients $\bar{x}_1, \dots, \bar{x}_n$, not all zero, and $b \in \mathbb{R}$ such that:

$$\sum_{i=1}^{n} \bar{x}_i u_i = \langle \bar{x}, u \rangle \leq b \leq \sum_{i=1}^{n} \bar{x}_i w_i = \langle \bar{x}, w \rangle,$$

for all $u = (u_1, \dots, u_n) \in Q_v$, $w = (w_1, \dots, w_n) \in C$. It is straightforward to observe the following facts:

1. All $\bar{x}_i$ must be nonnegative and, since they cannot be all zero, we can assume $\sum \bar{x}_i = 1$; suppose on the contrary there is i such that $\bar{x}_i < 0$. Since $x_n = (0, \dots, -n, \dots, 0) \in Q_v$, with $-n$ at the i-th place in the vector, it must be $\langle \bar{x}, x_n \rangle \leq b$, but $\langle \bar{x}, x_n \rangle \to +\infty$ if $n \to +\infty$, contradiction
2. $b = v$; First of all, since $\bar{v} := (v, \dots, v) \in Q_v$ we have, from $\langle \bar{x}, \bar{v} \rangle \leq v$ that $b \geq v$. Suppose now $b > v$, and take $a > 0$ so small that $b > v + a$. Then $\sup\{\sum_{i=1}^{n} \bar{x}_i u_i \ : \ u \in Q_{v+a}\} < b$, and this implies $Q_{v+a} \cap C = \emptyset$, against the definition of v
3. $Q_v \cap C \neq \emptyset$. On the contrary, suppose $Q_v \cap C = \emptyset$; this is equivalent to saying that $\max_i x_i > v$, for all $x \in C$. As $x \mapsto \max_i x_i$ is a continuous function, it assumes minimum, say $a > v$, on the compact set C. But then $Q_l \cap C = \emptyset$, for all $l \leq a$, and this contradicts the definition of v.

Let us consider now the inequality:

$$v \leq \sum_{i=1}^{n} \bar{x}_i w_i = \langle \bar{x}, w \rangle,$$

for $w \in C$. Given any $\beta \in S_m$, $w \stackrel{\text{def}}{=} \sum_{j=1}^{m} \beta_j p_j \in C$. Thus

$$f(\bar{x}, \beta) = \langle \bar{P}\beta, \bar{x} \rangle \sum_{i,j} \bar{x}_i \beta_j p_{ij} \geq v. \tag{6.1}$$

This means that the first player, by using $\bar{x}$ guarantees himself at least v, no matter what the second player does. Now, let $\bar{w} \in Q_v \cap C$ (see 3. above). Since $\bar{w} \in C$, then $\bar{w} = \sum_{j=1}^{m} \bar{\beta}_j p_j$, for some $S_m \ni \bar{\beta} = (\bar{\beta}_1, \dots, \bar{\beta}_m)$. Since $\bar{w} \in Q_v$, then $\bar{w}_i \leq v$ for all i. Thus, for all $\lambda \in S_n$, we get:

$$f(\lambda, \bar{\beta}) = \sum_{ij} \lambda_i \bar{\beta}_j p_{ij} = \langle P\bar{\beta}, \lambda \rangle \leq v. \tag{6.2}$$

Thus $(\bar{\alpha}, \bar{\beta})$ is a saddle point of the game and $v = f(\bar{\alpha}, \bar{\beta})$ is the value of the game. ■

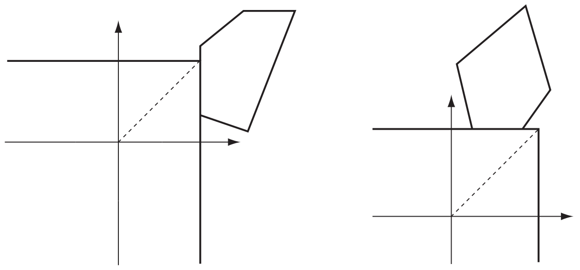
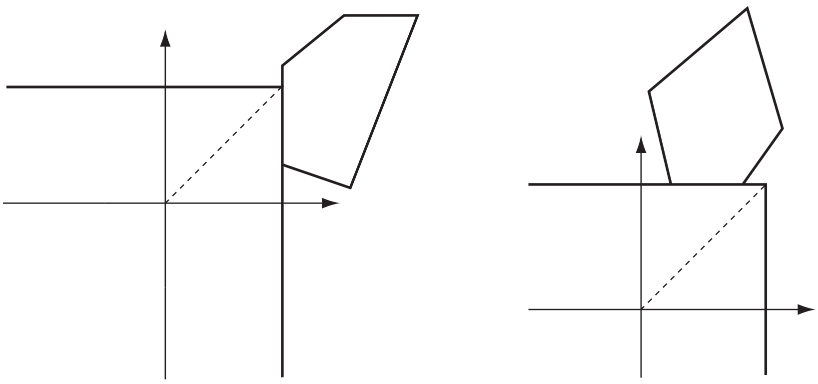


Fig. 6.5. In the Figure on the left the first row is optimal for the first player, in the right the second row is optimal for the first player



The first row is optimal for the first player *The second row is optimal for the first player*

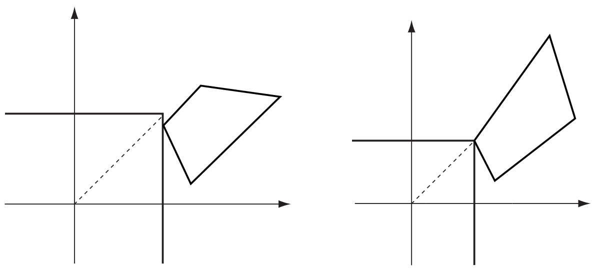


Fig. 6.6. In the Figure on the left a saddle point in pure strategies, what happens in the Figure on the left?

The above proof suggests, at least in principle, a way to solve the game: an optimal strategy for the first player is given by the (normalized) vector normal to a separating hyperplane, an optimal strategy for the second one can be obtained by considering a point lying in C and Q_v at the same time. Since the point lies in C, it is a convex combination of the columns of the matrix. The coefficients of this convex combination provide then an optimal strategy for the second player. This remark is however useful when one of the players has only two available strategies: if both of them have at least three strategies, the calculations are not simple.

To conclude this section, we prove an interesting result, which is a consequence of the Farkas lemma; it will be used in the sequel, and it has a very reasonable interpretation: it states that if a player, say the second, can never play a column at an equilibrium, the first one has a strategy giving him at least the value against any other column, and more than the value of the game against that column.

Proposition 6.2.1 *Given a game described by an $n \times m$ matrix P and with value v, either the second player has an optimal strategy $\bar{q} = (\bar{q}_1, \dots, \bar{q}_m)$ such that $\bar{q}_m > 0$, or the first player has an optimal strategy $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$ such that $\langle \bar{x}, p_m \rangle > v$, where p_m is the m^{th} column of the matrix P.*

Proof. Without loss of generality we can assume $v = 0$. Now consider the $n + m$ vectors

$$e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1), p_1, \dots, p_{m-1}, -p_m.$$

It can happen that $-p_m$ is in the convex cone C generated by the other vectors, or it is not. We shall show that in the first case the second player has an optimal strategy with last component positive, while in the second case the first player has an optimal strategy guaranteeing to him positive payoff against the last column. In the first case, there are nonnegative numbers $\rho_1, \dots, \rho_n, \lambda_1, \dots, \lambda_{m-1}$ such that

$$-p_m = \sum_{j=1}^{n} \rho_j e_j + \sum_{j=1}^{m-1} \lambda_j p_j.$$

This implies

$$\sum_{j=1}^{m-1} \lambda_j p_{ij} + p_{im} = -\rho_i \leq 0, \tag{6.3}$$

for all i. Setting $\bar{q}_j = \frac{\lambda_j}{1+\sum \lambda_i}$, $j = 1, \dots, m-1$, $\bar{q}_m = \frac{1}{1+\sum \lambda_i}$, $\bar{q} = (\bar{q}_1, \dots, \bar{q}_m)$, then (6.3) can be rewritten

$$\langle p_i, \bar{q} \rangle \leq 0,$$

for all (rows) i, and this shows that $\bar{q}$ is the optimal strategy for the second player we are looking for (remember: $v = 0$). Suppose now $-p_m \notin C$. Then there are numbers $\lambda_1, \dots, \lambda_n$ such that, setting $\lambda = (\lambda_1, \dots, \lambda_n)$:

$$\langle e_j, \lambda \rangle \geq 0, j = 1, \dots, n \quad \langle \lambda, p_j \rangle \geq 0, j = 1, \dots, m-1, \quad \langle -p_m, \lambda \rangle < 0.$$

The first inequality guarantees that $\lambda_i \geq 0$ for all i and the third one that they cannot be all zero. Setting $\bar{x}_i = \frac{\lambda_i}{\sum \lambda_i}$, $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$, we finally conclude that $\bar{x}$ is an optimal strategy for the first player with the required properties. ■

6.3 Linear Programming

As already observed, since finding optimal strategies in zero sum games is not easy, it is a good idea to translate the problems for the players in two linear programming problems. Conversely, the minimax theorem of von Neumann can be used to prove the fundamental theorems of linear programming. This is what we do now.

So, let us now introduce the linear programming problem.

Suppose we have an $m \times n$ matrix A and vectors b, c belonging to $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Then the following problem is a *linear programming* problem:

$$\begin{cases} \min\langle c, x\rangle \text{ such that} \\ x \geq 0, Ax \geq b \end{cases}. \tag{6.4}$$

It turns out that it is quite useful to associate to such a problem a "similar" one, which is called the *dual* problem, which is related to the first one. With a joint analysis of them, often one can obtain information that it is not possible to get from just one of the two problems. Let us now define the problem dual to the above one.

Definition 6.3.1 *Let A be an $m \times n$ matrix and let b, c be vectors belonging to $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. The following two linear programming problems are said to be in duality*:

$$\begin{cases} \min\langle c, x\rangle \text{ such that} \\ x \geq 0, Ax \geq b \end{cases}, \tag{6.5}$$

$$\begin{cases} \max\langle y, b\rangle \text{ such that} \\ y \geq 0, A^T y \leq c \end{cases}. \tag{6.6}$$

Sometimes it can happen that the linear problem has a different shape: for instance, in the primal problem the non negativity conditions on the variable could be missing (we shall see later an interesting example when this occurs). Thus, we see now how to dualize such a problem.

Theorem 6.3.1 *Let A be an $m \times n$ matrix and let b, c be vectors belonging to $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. The following two linear programming problems are in duality:*

$$\begin{cases} \min\langle c, x\rangle \text{ such that} \\ Ax \geq b \end{cases}, \tag{6.7}$$

$$\begin{cases} \max\langle y, b\rangle \text{ such that} \\ y \geq 0, A^T y = c \end{cases}. \tag{6.8}$$

Proof. We need to translate the problem (6.7) into an equivalent form (6.11) and then find its dual. To do this, let us use the following trick: set $u = (x, y)$ $\gamma = (c, -c)$, $\Gamma = [A, -A]$, $\beta = b$. The problem:

$$\begin{cases} \min\langle \gamma, u\rangle \text{ such that} \\ u \geq 0, \Gamma u \geq \beta \end{cases}, \tag{6.9}$$

is clearly equivalent to (6.7) (what we have done is to "decouple" the variable x in the difference between two positive numbers). Let us dualize it:

$$\begin{cases} \max\langle y, \beta\rangle \text{ such that} \\ y \geq 0, \Gamma^T y \leq \gamma \end{cases}. \tag{6.10}$$

To conclude, it is enough to observe that, if we look at i-th and $n+i$-th constraint equation in (6.10) they read as:

$$a_{1i}y_1 + \cdots + a_{mi}y_m \le c_i,$$

$$-a_{1i}y_1 - \cdots - a_{mi}y_m \le -c_i,$$

providing the required equality constraints in the problem (6.8). ■

Let us prove a first important result on problems in duality. Denote by v the value of the primal problem and by V the value of the dual problem. Then:

Proposition 6.3.1 *It always holds:*

$$v \ge V.$$

Proof. Let us see the result in both cases, since it is a very easy calculation. In the first case:

$$\langle c, x\rangle \ge \langle x, A^t y\rangle = \langle Ax, y\rangle \ge \langle b, y\rangle.$$

This being true for all admissible x and y provides the result in the first case. In the second, observe that the same string of inequalities holds, with the difference that the first inequality actually is an equality.

The above result is important and easy to show. There is, however, a deeper connection between two problems in duality. We now prove the fundamental results in this setting.

To describe the result, first of all let us agree to call *feasible (unfeasible)* a problem such that the constraint set is nonempty (empty), and unbounded the minimum (maximum) problem if its value is $-\infty$ (∞).

Then the following results hold:

Theorem 6.3.2 *Suppose the linear programming problem $\mathcal{P}$ is feasible, and its dual problem is unfeasible. Then the problem $\mathcal{P}$ is unbounded.*

Proof. Since the constraint set is a polyhedral set, what we need to do is to find a direction, of the form $\hat{x} + tx$, $t \ge 0$, along which the objective function $\langle c, \cdot\rangle$ goes to $-\infty$, equivalently $\langle c, x\rangle < 0$. In order to find the direction as above, we should find x such that $Ax \ge 0$ and $\langle c, x\rangle < 0$, since by feasibility there is $\hat{x}$ such that $A\bar{x} \ge b$, and thus $\hat{x} + tx$ fulfills the constraints. To do it, we appeal to a suitable game. Let us now see the details.

Let us consider the game described by the following matrix:

$$\begin{pmatrix} a_{11} & \dots & a_{m1} & -c_1 \\ \dots & \dots & \dots & \dots \\ a_{1n} & \dots & a_{mn} & -c_n \end{pmatrix} = \left(A^t | c\right).$$

Step 1. Let us see first that this game has value $v \ge 0$. Otherwise there would be a strategy $q = (q_1, \dots, q_m, q_{m+1})$ for the second player such that it guarantees to her to get a negative quantity against each row chosen by the first player. In formulas:

$$a_{1j}q_1 + \cdots + a_{mj}q_m - c_j q_{m+1} < 0 \quad j = 1, \dots, n.$$

This will lead to a contradiction: for, if $q_{m+1} > 0$, setting $z_i = \frac{q_i}{q_{m+1}}$, $z = (z_1, \dots, z_m)$, we get that

$$A^T z < c, \qquad z \geq 0,$$

against the assumption that the dual problem is unfeasible. On the other hand, if $q_{m+1} = 0$, this implies that, calling $z = (q_1, \dots, q_m)$, then $A^T z \ll 0$ (the notation $a \ll b$ means $a_i < b_i$ for all i). But then, for a sufficiently large k, kz is feasible for the dual problem. Impossible.

Step 2. We see now that, if the value of the game is zero, then necessarily, for any optimal strategy $q = (q_1, \dots, q_m, q_{m+1})$ of the second player, it must be $q_{m+1} = 0$. Otherwise, with a similar argument as before, we see that

$$a_{1j}q_1 + \dots + a_{mj}q_m - c_j q_{m+1} \leq 0 \quad j = 1, \dots, n,$$

and, setting $z_i = \frac{q_i}{q_{m+1}}$, $z = (z_1, \dots, z_m)$, we get that

$$A^T z \leq c, \qquad z \geq 0,$$

and this is impossible.

Step 3. Let us now consider the first player. I claim that he has a strategy $x = (x_1, \dots, x_n)$ such that

$$Ax \geq 0, \quad \langle x, c \rangle < 0.$$

This is immediate if the value of the game is positive, as he will be able to get a positive payoff against each column. On the other hand, if the value of the game is 0, from the previous step we know that the second player cannot use the last column. Then the claim follows from Proposition 6.2.1.

Step 4. Now we are able to conclude. Since the minimum problem is feasible, there exists $\hat{x}$ such that $\hat{x} \geq 0$ and $A\hat{x} \geq b$. Consider now $x_t = \hat{x} + tx$, $t \geq 0$. Clearly, x_t verifies $x_t \geq 0$ and $Ax_t \geq b$, for all $t > 0$. And from $\langle c, x \rangle < 0$ we get that $\langle c, x_t \rangle \to -\infty$, so that the problem is unbounded, and this ends the proof. ■

Let us now see what happens when both problems are feasible: this is the fundamental duality theorem in linear programming.

Theorem 6.3.3 *Suppose the two problems are both feasible. Then there are solutions $\bar{x}, \bar{y}$ of the two problems, and $\langle c, \bar{x} \rangle = \langle b, \bar{y} \rangle$.*

Proof. Again, we prove the theorem by appealing to a suitable game. Consider the following $(m + n + 1)$ square matrix:

$$\begin{pmatrix} 0 & \dots & 0 & -a_{11} & \dots & -a_{1n} & b_1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -a_{m1} & \dots & -a_{mn} & b_m \\ a_{11} & \dots & a_{m1} & 0 & \dots & 0 & -c_1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{1n} & \dots & a_{mn} & 0 & \dots & 0 & -c_n \\ -b_1 & \dots & -b_m & c_1 & \dots & c_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & -A & b \\ A^T & 0 & -c \\ -b & c & 0 \end{pmatrix}.$$

Observe that the above matrix is skew symmetric, and thus its value is zero and the optimal strategies for the players are the same. Let us call $(p, q, t) = (p_1, \dots, p_m, q_1, \dots, q_n, t)$

an optimal strategy for the first player. He will get a nonnegative payoff, by playing the above strategy, against any column chosen by the second player. Thus:

$$Aq - tb \geq 0 \quad - A^T p + tc \geq 0 \quad \langle p, b\rangle - \langle q, c\rangle \geq 0.$$

From this, the result easily follows, provided $t > 0$. For, setting $\bar{x} = \frac{q}{t}$, $\bar{y} = \frac{p}{t}$ from the above relations we get:

$$A\bar{x} \geq b \quad A^T\bar{y} \leq c \quad \langle \bar{y}, b\rangle \geq \langle \bar{x}, c\rangle.$$

But the first two conditions just say that $\bar{x}$ and $\bar{y}$ are feasible for the problem and its dual respectively, while the third one is the required optimality condition, just remembering that the opposite inequality must hold at every pair of feasible vectors. And this explains why it is a good idea to consider the above matrix. So, let us suppose by contradiction $t = 0$, for every optimal strategy for the first player. In such a case, there must be an optimal strategy $(p_1, \dots, p_m, q_1, \dots, q_m, 0)$ for the second player guaranteeing her to get a strictly negative result against the last row (Proposition 6.2.1).[4] Moreover, at every optimal strategy of the second player, she will play the last column with probability zero, because the first one plays the last row with probability zero. This amounts to saying that:

$$-Aq \leq 0 \quad A^T p \leq 0 \quad - \langle b, p\rangle + \langle c, q\rangle < 0.$$

Since both problems are feasible, there are $\hat{p} \geq 0, \hat{q} \geq 0$, such that $A\hat{q} \geq b$, $A^T\hat{p} \leq c$. As $\langle c, q\rangle < \langle b, p\rangle$, if $\langle c, q\rangle < 0$, then $\langle c, \hat{q} + rq\rangle \to -\infty$, for $r \to \infty$, but this is impossible, as the dual problem is feasible. Thus $\langle c, q\rangle \geq 0$, and so $\langle b, p\rangle > 0$. Again this leads to a contradiction, because it would imply that the dual problem is unbounded, against the assumption that the primal problem is feasible. Thus it must be $t > 0$, for at least an optimal strategy for the first player. This concludes the proof. ∎

Summarizing the previous results, we have seen that if one problem is feasible and the other one is not, then necessarily the feasible one is unbounded; on the other hand, if both are feasible, then they both have solutions and there is no duality gap. Equivalently, if we know that one problem is feasible and has solution (for instance from a compactness argument), then the dual is feasible, has solution and there is no duality gap.

6.4 A direct existence proof for correlated equilibria, using a duality argument

As we have already mentioned, to prove that for any bimatrix (A, B) the set of the correlated equilibria is nonempty it is enough to observe that a mixed equilibrium is also a correlated equilibrium. Then the Nash theorem does the job. However, this proof is not completely satisfactory, since it is indirect, and it finally relies on the use of Kakutani theorem. Since by definition a vector is a correlated equilibrium if it satisfies a certain number of linear inequalities, it is natural to ask the question

[4] Since the game is symmetric an optimal strategy for the second player cannot have positive probability allocated on the last column.

whether it is possible to use linear programming to prove the existence result. This is the content of this section.

Given an $n \times m$ bimatrix (A, B), a vector $p = (p_{ij})$, $i = 1, \dots, n$, $j = 1, \dots, m$ is a correlated equilibrium if it lies in the nm-simplex and satisfies the inequality constraint system:

$$Cx \geq 0,$$

where C is the following $(n^2 + m^2) \times nm$ matrix

$$C(A,B) = \begin{pmatrix}
a_{11}-a_{11} & \cdots & a_{1m}-a_{1m} & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{11}-a_{n1} & \cdots & a_{1m}-a_{nm} & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \cdots & \cdots & a_{n1}-a_{11} & \cdots & a_{nm}-a_{1m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \cdots & \cdots & a_{n1}-a_{n1} & \cdots & a_{nm}-a_{nm} \\
b_{11}-b_{11} & \cdots & 0 & \cdots & \cdots & \cdots & b_{n1}-b_{n1} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
b_{11}-b_{1m} & \cdots & 0 & \cdots & \cdots & \cdots & b_{n1}-b_{nm} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdot & b_{1m}-b_{11} & \cdot & \cdot & \cdot & 0 & \cdot & b_{nm}-b_{n1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdot & b_{1m}-b_{1m} & 0 & 0 & \cdots & 0 & 0 & b_{nm}-b_{nm}
\end{pmatrix}.$$

Now, consider the following linear programming problem:

$$\begin{cases} \inf\left(-\sum p_{ij}\right) \text{ such that} \\ p \geq 0, Cp \geq 0 \end{cases}, \tag{6.11}$$

It is quite clear that this problem is feasible, since $p = 0$ satisfies the constraints. Moreover, it is also clear that the existence of a correlated equilibrium is equivalent to saying that the above LP problem admits a non null feasible vector p, and this in turn is equivalent to saying that the problem is (lower) unbounded, since tp is still a solution, for all $t > 0$. Unboundedness however is equivalent in this case to the fact that the dual problem is unfeasible.

The dual problem is:

$$\begin{cases} \sup\langle 0, \lambda\rangle \text{ such that} \\ \lambda \geq 0, C^T\lambda \leq -1 \end{cases}, \tag{6.12}$$

where $\lambda = (\lambda^1_{de}, \lambda^2_{gh})$, with $d, e = 1, \dots, n$, $g, h = 1, \dots, m$.

Thus the statement will we proved once we show that this problem is unfeasible. To see this, it is clearly enough to show that for every λ non negative dual variable it exists $x \in \mathbb{R}^{nm}$ such that $x \geq 0$, $x \neq 0$ and $xC^T\lambda \geq 0$. We shall see that, quite naturally, it is possible to take x of the form $x = x_{ij} = u_i v_j$, with $u = (u_i)$, $v = (v_j)$

in the simplexes of dimension n and m respectively. The key point is to evaluate the coefficient of the terms a_{ij} and b_{ij} in the product $xC^T\lambda$. We shall see this for the coefficients a_{ij} relative to the first player. Thus, we need to consider only $\lambda^1_{de} = \lambda$, and we shall more simply write λ in the form[5]

$$\lambda = (\lambda_{11}, \dots, \lambda_{1n}, \lambda_{21}, \dots, \lambda_{2n}, \dots, \lambda_{n1}, \dots, \lambda_{nn}).$$

the coefficient a_{ij} appears n times with a positive sign, in correspondence of the row i against all other rows (row ij, columns from $n(i-1)+1$ to ni included) and with negative coefficient once for every row against row i (in the rows $i, i+m, \dots i+(n-1)m$, columns $i, 2i, ni$). Writing $x = (u_1v_1, u_1v_2, \dots, u_2v_1, \dots, u_nv_m)$, it finally turns out that the coefficient relative to a_{ij} is the following:

$$v_j(u_i \sum_k \lambda_{ik} - \sum_j u_j\lambda_{ji}).$$

Thus, the thesis follows once we show existence of a vector u in the n simplex solving the system of linear equations:

$$u_i \sum_k \lambda_{ik} = \sum_j u_j\lambda_{ji}. \tag{6.13}$$

This is an immediate consequence of the next lemma, and this end the proof. ■

Lemma 1. *Suppose $A = (a_{ij})$ is a $n \times n$ matrix such that $a_{ij} \geq 0$ for all i and j and $a_{ii} = 0$. Then the system:*

$$\begin{cases} x_1 \sum_k a_{1k} = \sum_j x_j a_{j1} \\ \dots \\ x_n \sum_k a_{nk} = \sum_j x_j a_{jn} \end{cases}, \tag{6.14}$$

has a solution $x = (x_1, \dots, x_n)$ such that $x_i \geq 0$ and $x \neq 0$.

Proof. Suppose at first that $a_{ki} > 0$ for all k, i such that $k \neq i$. Writing the system in the form $Bx = 0$, it holds that $\sum_i b_{ij} = 0$ for all j, thus the system is singular and it admits a non null solution. Now let x be one solution having (at least) $n \geq l \geq 1$ positive components (such a solution clearly exists). If $l = n$ there is nothing to prove. Then suppose $l < n$ and, without loss of generality, $x_i > 0$ for $i = 1, \dots, l$. Consider the first l equations:

$$\begin{cases} (a_{11} + \dots + a_{n1})x_1 - a_{21}x_2 - \dots a_{l1}x_l - a_{(l+1)1}x_{l+1} - \dots - a_{n1}x_n = 0 \\ -a_{12}x_1 + (a_{12} + \dots + a_{n2})x_2 - \dots a_{l2}x_l - a_{(l+1)2}x_{l+1} - \dots - a_{n2}x_n = 0 \\ \dots \\ -a_{1l}x_1 - a_{2l}x_2 + \dots (a_{1l} + \dots + a_{nl})x_l - a_{(l+1)l}x_{l+1} - \dots - a_{nl}x_n = 0 \end{cases} \tag{6.15}$$

summing them up we get:

$$(\sum_{k>l} a_{k1})x_1 + \dots + (\sum_{k>l} a_{kl})x_l = \sum_{k=1}^{l} \sum_{j>l} a_{kj}x_j.$$

[5] To better memorize, let think of the coefficient ij as "row i against row j".

Since the left size of the above equality is a positive number, it follows that at least one x_j, $j > l$, must be positive, and this shows the claim in the particular case $a_{ki} > 0$ for all k, i such that $k \neq i$. Now consider the general case with matrix A, and associate to it the family of matrices $A_\varepsilon = (a^\varepsilon_{ij})$, where $a^\varepsilon_{ij} = a_{ij} + \varepsilon$ if $i \neq j$, $a^\varepsilon_{ii} = 0$. Since A_ε fulfills the assumptions of the previous case, it exists a solution x_ε, for all $\varepsilon > 0$, in the simplex. Every limit point of $\{x_\varepsilon, \varepsilon > 0\}$ is a required solution for the original system, and this completes the proof of the lemma. ■

6.5 A different algorithm to solve zero sum games

The results I present here want to characterize the extreme optimal strategies of the players in a zero sum game. From this characterization we shall see that it is possible to find these strategies by solving (several!) linear systems. The number of these system grows explosively with the dimension of the matrix of the game, but there are powerful and simple software packages able to solve those systems.

The basic idea is that any optimal strategy solves a number of equalities and a number of inequalities, but the extreme ones have the property that they are the unique solution of a certain system of equalities. We present the results focussing the attention on the strategies of the second player, the same can be of course done for the first player.

Denote by

$$P = \begin{pmatrix} p_{11} & \cdots & p_{1m} \\ \cdots & p_{ij} & \cdots \\ p_{n1} & \cdots & p_{nm} \end{pmatrix}$$

the matrix of the game. An optimal strategy y for the second player must verify equalities/inequalities of the form:[6]

$$\begin{cases} p_{11}y_1 + \dots p_{1m}y_m = v \\ \cdots \quad \cdots \quad \cdots \\ p_{l1}y_1 + \dots p_{lm}y_m = v \\ p_{(l+1)1}y_1 + \dots p_{(l+1)m}y_m < v \\ \cdots \quad \cdots \quad \cdots \\ p_{n1}y_1 + \dots p_{nm}y_m < v, \end{cases} \tag{6.16}$$

for some $l \in \{1, \dots, n\}$.

The following theorem is the basic tool to construct the algorithm for finding extremal optimal strategies.

Theorem 6.5.1 *Let $y = (y_1, \dots, y_s, 0, \dots, 0)$ be an optimal strategy (for the second player). Then y is an extremal optimal strategy if and only if there is $l \geq 1$ such that $(y_1, \dots, y_s, v)$ is the unique $s+1$-dimensional solution of the following system:*

[6] From now on, we will renumber rows/columns of the matrix P anytime is necessary in order to have simpler notations.

$$\begin{cases} p_{11}x_1 + \dots p_{1s}x_s - x_{s+1} = 0 \\ \dots \quad \dots \quad \dots \\ p_{l1}x_1 + \dots p_{ls}x_s - x_{s+1} = 0 \\ x_1 + \dots x_s = 1. \end{cases} \tag{6.17}$$

Proof. Suppose y is not an extremal optimal solution, and write the system of all equalities satisfied by the non null components of y. There are y^1, y^2 optimal solutions such that $y = \frac{1}{2}(y^1 + y^2)$. Then both y^1 and y^2 are solutions of the system (6.17): first of all, since $y_k = 0$ for $k > l$ and y^1_k, y^2_k are non negative for the same k, it follows that actually $y^1_k = 0$, $y^2_k = 0$, for $k > l$, showing that the last equation is verified by both. Next, since y^1 and y^2 are optimal strategies, it holds that:

$$\sum_{j=1}^{s} p_{ij} y^k_j \leq v, \tag{6.18}$$

for $k = 1, 2$ and $i = 1, \dots, l$. Since $y = \frac{1}{2}(y^1 + y^2)$ verifies equalities in (6.18), it follows that also y^1 and y^2 verify equalities in (6.18). Thus $(y^1_1, \dots, v)$ and $(y^2_1, \dots, v)$ are distinct solutions of (6.17). Conversely, suppose y is an extremal solution, and construct the system of linear equalities (6.17) solved by $(y_1, \dots, y_s, v)$. We want to prove that this system has unique solution. Suppose instead it has to distinct solutions, say (y^1, v^1) and (y^2, v^2). Consider the following two m-dimensional vectors:

$$\hat{y}^1 = (y_1 + \varepsilon(y^1_1 - y^2_1), \dots, y_s + \varepsilon(y^1_s - y^2_s), 0, \dots, 0),$$
$$\hat{y}^2 = (y_1 - \varepsilon(y^1_1 - y^2_1), \dots, y_s - \varepsilon(y^1_s - y^2_s), 0, \dots, 0).$$

It is easy to verify that both vectors do represent strategies for the second player, provided ε is small enough. Now, suppose $v^1 \leq v^2$. We prove that in this case $\hat{y}^1$ is an optimal strategy. For, if $i = 1, \dots, l$, then

$$\sum_{j=1}^{s} p_{ij} \tilde{y}^1_j = v + \varepsilon(v^1 - v^2) \leq v, \tag{6.19}$$

while if $i > l$, then

$$\sum_{j=1}^{s} p_{ij} \tilde{y}^1_j < v + \varepsilon(v^1 - v^2) \leq v. \tag{6.20}$$

It follows that $\hat{y}^1$ is optimal, and also that $v^1 = v^2$; otherwise in (6.19) and (6.20) there would be strict inequality for all i, and this would imply that the second player could pay less that v against all rows, and this is impossible, since v is the value of the game. But this in turn implies that also $\hat{y}^2$ is optimal. Since $\hat{y}^1 \neq \hat{y}^2$ (as $v^1 = v^2$), and $\tilde{y} = \frac{1}{2}(\hat{y}^1 + \hat{y}^2)$, it follows that y is not extremal, and this contradiction ends the proof. ■

Remark 6.5.1 Consider the following matrix:

$$\begin{pmatrix} 0 & 2 & -4 & 0 \\ -2 & 0 & 0 & 4 \\ 4 & 0 & 0 & -6 \\ 0 & -4 & 6 & 0 \end{pmatrix}.$$

It is easy to show that $(0, \frac{3}{5}, \frac{2}{5}, 0)$ is an optimal extremal strategy (observe that the game is symmetric so that the value is $v = 0$). Observe that the system (6.17) associated to this solution is *not* a square system, nevertheless, as expected, it has only one solution. Of course in any case it must be, in Theorem 6.5.1, $l \geq s$.

Now, a deeper analysis of the structure of a system like that one in (6.17) allows us making precise a procedure to find the extremal optimal solutions.

Theorem 6.5.2 *Suppose y is an extremal optimal strategy for the second player. Then there is a square submatrix $\hat{P}$ of P such that*

$$y_j = \frac{\sum_i \hat{P}_{ij}}{\sum_{ij} \hat{P}_{ij}}, \qquad v = \frac{|\hat{P}|}{\sum_{ij} \hat{P}_{ij}}, \tag{6.21}$$

where $\hat{P}_{ij}$ is the cofactor of p_{ij} in the matrix $\hat{P}$.[7] *Analogously for the first player, if x is an extremal optimal strategy for the first player, there is a square submatrix $\tilde{P}$ of P such that*

$$x_i = \frac{\sum_j \tilde{P}_{ij}}{\sum_{ij} \tilde{P}_{ij}}. \tag{6.22}$$

Proof. First of all, observe that there must me a square subsystem of the system (6.17), including last equation, and equivalent to the whole system. The rest of the proof, is just a consequence of Cramer's rule for solving the square system (6.17). The interested reader could fully write down the calculation for instance in the case $\hat{P}$ is a 3×3 matrix. As natural, the determinant of the matrix of the coefficients

$$\begin{pmatrix} p_{11} & \dots & p_{1s} & -1 \\ \vdots & \vdots & & \\ p_{s1} & \dots & p_{ss} & -1 \\ 1 & \dots & 1 & 0 \end{pmatrix}$$

should be calculated by expanding along the last row, and the resulting cofactors calculated by expanding along the last column. ∎

Thus, an algorithm to find extremal optimal solutions will examine all square submatrices of A to compute all possible extreme optimal strategies and possible values. All x and y optimal strategies with correct v calculated by (6.17) are extreme optimal strategies. All optimal strategies are convex combination of extremal optimal strategies.

To clarify the above procedure, let us see some examples.

Example 6.5.1 Let the game be described by the matrix:

$$\begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix}.$$

In this case is very easy to see that the extreme optimal strategies for the players are $(1, 0)$ (unique) for the first and $(1, 0)$ and $(\frac{1}{2}, \frac{1}{2})$ for the second. Let us now apply

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[7] In the case $\hat{P}$ is just the entry p_{ij}, the preceding result applies in the sense that $y_j = 1$ and $v = p_{ij}$.

the algorithm. Let us start by the (degenerate) matrix p_{11}. It provides $(1, 0)$ for both players, with $v = 2$. For sure this is not an optimal strategy for the second player, since it does *not* guarantee to her the value against the second row. It guarantees the value against both columns, thus it is a candidate for being an extremal optimal strategy for the first player. The coefficient p_{12} instead is a candidate for both players, and thus it is an equilibrium indeed (since now we can be sure that the value is 2) (this implies also that the candidate found before is actually optimal for the first player, but this does not add information). It remain to consider the matrix made by the whole game. We have that the matrix of the cofactors is

$$\begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}.$$

The determinant of the matrix is -4, $\sum_{ij} p_{ij} = -2$, for player I we get $x_1 = (1, 0)$, for player II we get $(\frac{1}{2}, \frac{1}{2})$, moreover $v = 2$ and we conclude as expected.

Example 6.5.2 Let the game be described by the matrix:

$$\begin{pmatrix} 5 & 5 & 20 \\ 6 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

Consider the matrix itself. Its cofactor matrix is given by

$$\begin{pmatrix} 40 & -48 & 0 \\ -40 & 40 & 0 \\ -100 & 120 & -5 \end{pmatrix}$$

Since $x_1 = \frac{40-48+0}{40-48+0-40+40+0-100+120-5} = \frac{-8}{7}$, $y_3 = \frac{-5}{7}$, we see that the matrix cannot provide a strategy for the players. Consider now all the 2×2 submatrices (with non vanishing determinant):

$$\begin{pmatrix} 5 & 5 \\ 6 & 5 \end{pmatrix}, \begin{pmatrix} 5 & 20 \\ 6 & 0 \end{pmatrix}, \begin{pmatrix} 5 & 20 \\ 5 & 0 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 20 & 8 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 8 \end{pmatrix}.$$

The first one has cofactor matrix:

$$\begin{pmatrix} 5 & -6 \\ -5 & 5 \end{pmatrix}.$$

It follows that $x = (\frac{5-6}{5-6+5-5}, \frac{-5+5}{5-6+5-5}, 0) = (1, 0, 0)$, $y = (0, 1, 0)$, $v = \frac{-5}{5-6+5-5} = 5$. A direct calculation shows that the (pure) strategy so found is an equilibrium strategy. Thus we now know that the value of the game is 5. Ignoring for the moment this information, we see that the second submatrix provides $x = (\frac{2}{7}, 0, \frac{5}{7})$, $y = (\frac{20}{21}, \frac{1}{21}, 0)$, $v = \frac{-120}{-21}$. However this pair cannot be optimal since player one can guarantees himself only $\frac{10}{7}$, player two $\frac{125}{21}$, both not agreeing with v. Performing all calculations, we see that there is only one more submatrix providing extremal strategies,

$$\begin{pmatrix} 5 & 20 \\ 5 & 0 \end{pmatrix},$$

since it offers $x = (\frac{1}{4}, \frac{3}{4}, 0)$, $y = (0, 1, 0)$, $v = \frac{-100}{-20} = 5$.

Finally, observe that the above calculations can be simplified whenever we know the value of the game. This happens, for instance, when there is an equilibrium in pure strategies (which is easy to be detected, whenever it exists, at least in small matrices), or also when the game is symmetric, so that the value is zero.

6.6 Exercises

Exercise 6.6.1 Given the zero sum game described by the following matrix:

$$\begin{pmatrix} a & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

1. find the conservative values for the players depending on $a \in \mathbb{R}$
2. say for which a there are equilibria in pure strategies
3. for $a = 1/2$ find the equilibrium of the game knowing that the first player never plays the second strategy
4. for $a > 1$ find the best reply of the second player to the strategy $(p, 0, 1-p)$ of the first.

Solution

1. For the first player, the min on the first row is 1 if $a \geq 1$ and 0 otherwise, the min on the second and on the third row is 0. Then the conservative value for the first player is 0 if $a \leq 0$; a if $0 < a < 1$, 1 if $a \geq 1$.
   The max on the first column, for the second player, is a is $a > 1$ and 1 if $a \leq 1$, on the second column is 2 and on the third one is 1. Then the conservative value for the second player is 1 for each a
2. There are equilibria in pure strategies if and only if the two conservative values coincide, i.e. if $v_I = v_{II} = 1$. This happens when $a \geq 1$
3. Knowing that the first player never plays the second strategy, the matrix reduces to
   $$\begin{pmatrix} a & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$
   We can observe that the third column is now dominated by a convex combination of the first two columns. Writing the strategy of the first player as $(p, 0, 1-p)$, p is determined by observing that he can obtain $1 - (1/2)p$ from the first column and $2p$ from the second. There are no equilibria in pure strategies for the second player, then for the indifference principle we get $1 - (1/2)p = 2p$ which implies $p = 2/5$. Writing the strategy of the second player as $(q, 1-q, 0)$, by the indifference principle and noticing there are no equilibria in pure strategies for the first player, we get $1/2q + 2(1-q) = q$ which implies $q = 4/5$. The equilibrium is then $\left(\left(\frac{2}{5}, 0, \frac{3}{5}\right), \left(\frac{4}{5}, \frac{1}{5}, 0\right)\right)$
4. second player can get $ap + 1 - p$ from the first column, $2p$ from the second, 1 from the third one. The first column is dominated from the third one, as $a > 1$. The best reply is to play the second column if $p < 1/2$, to play the third one if $p > 1/2$ and to play a combination of the second and of the third if $p = 1/2$.

Exercise 6.6.2 Given the zero sum game described by the following matrix A:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & a \end{pmatrix},$$

1. find the conservative values of the players for all $a > 0$;
2. say when there is an equilibrium in pure strategies;
3. find all optimal strategies of the first player for all $a > 0$.

Solution

1. The conservative value for the first player is
$$2 \quad \text{if } a \geq 2, \qquad a \quad \text{if } 1 < a < 2, \qquad 1 \quad \text{otherwise.}$$
The conservative value for the second player is 2 $\forall a > 0$;
2. there is equilibrium in pure strategies when the conservative values of the players coincide, then when $a \geq 2$;
3. Suppose $(p, 1-p)$ is the strategy used by the first player. Then she gets, from the first column, $4 - 3p$, from the second one 2 and from the third one $a + (3-a)p$. So that the first one has to maximize
$$\min\{4 - 3p, 2, a + (3-a)p\}.$$
From the picture below (Fig. 6.7), where the cases $a = 0$, $a = 2$ and generic $0 < a < 2$ are drawn, it is seen that for $a \geq 0$ the extremal strategies for the first player are given by $\left(\frac{2}{3}, \frac{1}{3}\right)$ and $\left(\frac{2-a}{3-a}, \frac{1}{3-a}\right)$. In particular, when $a \geq 2$ we find the pure strategy $(1, 0)$.

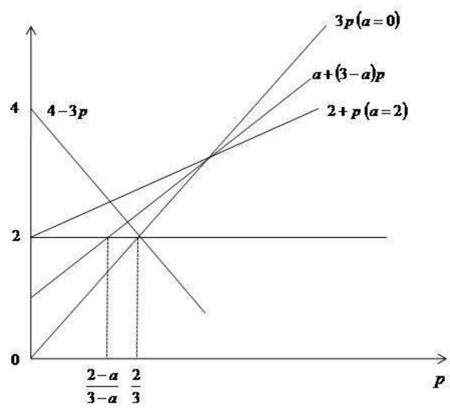

Fig. 6.7. Exercise 6.6.2

Exercise 6.6.3 Two players show at the same time the fingers of one hand (at least one) and say a number. They have to try to say the sum of the two numbers. If they both win or fail, they take zero, otherwise the winner wins the number he said.

1. say if there is equilibrium in pure strategies;
2. find the value of the game;
3. find the minimal dimension of the matrix representing the game;
4. write the matrix and find the optimal strategies when they can only show 1 or 3 fingers..

Solution

1. there are no equilibria in pure strategies, as the value is zero because of the symmetry but any strategy can guarantee at least zero against the strategies of the other player;
2. zero, because of the symmetry;
3. a matrix 25×25: for each one of the 5 choices of the first player of how many fingers to show, the 5 possible choices of the other player must be considered;
4. The matrix is:
$$\begin{pmatrix} 0 & 2 & -4 & 0 \\ -2 & 0 & 0 & 4 \\ 4 & 0 & 0 & -6 \\ 0 & -4 & 6 & 0 \end{pmatrix},$$
from which we get that the optimal strategies for the first player satisfy the system:
$$\begin{cases} -x_2 + 2x_3 \geq 0 \\ x_1 - 2x_4 \geq 0 \\ -2x_1 + 3x_4 \geq 0 \\ 2x_2 - 3x_3 \geq 0. \end{cases}.$$
as we already know that the value of the game is zero.
From the second and the third inequalities we easily get $x_1 = x_4 = 0$. From the first and the fourth ones we have, calling $x_2 = x$ and $x_3 = 1 - x$
$$\begin{cases} -x + 2 - 2x \geq 0 \\ 2x - 3 + 3x \geq 0. \end{cases}.$$
Then the optimal strategies for the first player (and for the second player, as the game is symmetric) are $(0, x, 1-x, 0)$, with $3/5 \leq x \leq 2/3$.

Exercise 6.6.4 Given the zero sum game described by the following matrix:
$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
find all the equilibria.

Solution The conservative value for the first player is $v_I = 0$ and for the second player is $v_{II} = 1$. As $v_I \neq v_{II}$ there are no equilibria in pure strategies.
We find the mixed strategies for the first player solving
$$\max v$$

$$\begin{cases} x_1 \geq v \\ x_2 \geq v \\ -x_1 + x_3 \geq v \\ x_1 + x_2 + x_3 = 1. \end{cases}.$$

then

$$\max v$$

$$\begin{cases} x_1 \geq v \\ x_2 \geq v \\ x_3 \geq 2v \\ x_1 + x_2 + x_3 = 1. \end{cases}.$$

from which we get $v = \frac{1}{4}$ and $(x_1, x_2, x_3) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$. We find the mixed strategies for the second player solving

$$\min v$$

$$\begin{cases} y_1 - y_3 \leq v \\ y_2 \leq v \\ y_3 \leq v. \end{cases}.$$

from which we get $(y_1, y_2, y_3) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$.

Exercise 6.6.5 Given the zero sum game described by the following matrix:

$$\begin{pmatrix} 4 & a & 2 \\ 2 & a & 4 \end{pmatrix}$$

find all the equilibria and the value of the game when $a \geq 0$.

Solution Using the idea of the proof of von Neumann theorem, we represent the pure strategies of the second player. If $a > 3$, II the second column is dominated and II plays a combination of the first and of the third column, the value of the game is $v = 3$; if $a = 3$ she plays a combination of all the three columns and $v = 3$. In both the cases, I plays the strategy $\left(\frac{1}{2}, \frac{1}{2}\right)$.
If $a < 3$ II plays the second column and $v = a$. To find the strategy of I we impose that to play the second column for the second player is the best reply to the strategy $(p, 1-p)$ of I. II pays $2p + 2$ playing the first column, a playing the second one and $-2p + 4$ playing the third one.We have to impose that

$$\begin{cases} a < 2p + 2 \\ a < 4 - 2p. \end{cases}.$$

When $2 < a < 3$ playing the second column is the best reply if $\frac{a-2}{2} < p < \frac{4-a}{2}$ and when $0 \leq a \leq 2$ if $0 \leq p \leq \frac{4-a}{2}$.

Exercise 6.6.6 Let us be given a nonempty closed convex set C in $\mathbb{R}^l$ $x \notin C$, and denote by p the projection of x over C. Prove that $y = p$ if and only if $y \in C$ and

$$\langle x - y, c - y \rangle \leq 0 \ \ \forall c \in C.$$

Proof. Outline. If $y \in C$ and $\langle x - y, c - y \rangle \leq 0 \ \forall c \in C$, then it is very easy to see that $\|x - y\| \leq \|x - c\|$, $\forall c \in C$, showing that y is actually the projection of x over C. Conversely, observe that

$$\|x - p\|^2 \leq \|x - [p + t(c - p)]\|^2,$$

for all $t \in (0, 1)$ and $c \in C$. Now develop calculations, simplify and let t go to 0^+.

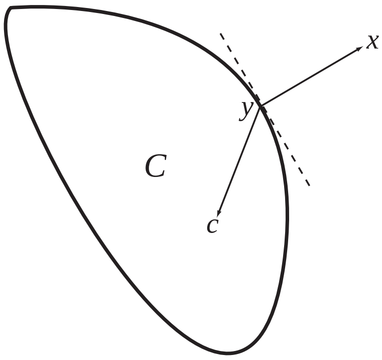

Fig. 6.8. The projection y of x on the set C

Exercise 6.6.7 Let A be a $2n \times 2n$ matrix representing a fair game, and suppose det $A \neq 0$. Then no optimal strategy can use all pure strategies.

Exercise 6.6.8 Consider the following game. Alex and Rosa must say, at the same time, a number between 1 and 4 (included). The one saying the highest number get from the other what he said. With one exception, otherwise the game is silly. If Alex says n and Rosa $n - 1$, then Rosa wins n. And conversely, of course. Write down the matrix associated to the game, find its value and its saddle points.

Exercise 6.6.9 Given the matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix},$$

1. find all equilibria in pure strategies;
2. find the best reply of the second player to the mixed strategy $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$;
3. find the value of the game;
4. find all equilibrium mixed strategies of the players.

Exercise 6.6.10 Given the matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ a & 0 & b \end{pmatrix},$$

with a, b real numbers:

1. find all a, b such that there are equilibria in pure strategies;

2. find all equilibria in mixed strategies for $a < 0$, $b < 0$;
3. find all equilibria in mixed strategies for $a = 1$, $b = 1$.

Exercise 6.6.11 Solve the following zero sum game:

$$\begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}.$$

Exercise 6.6.12 Solve the following zero sum game:

$$\begin{pmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{pmatrix}.$$

Exercise 6.6.13 Given the matrix:

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & b \end{pmatrix},$$

1. find all a, b such that there are equilibria in pure strategies;
2. find the value and all equilibria in mixed strategies for $b < 0$;
3. find the value and all equilibria in mixed strategies for $a = -1$, $b = 1$.

7

Bayesian Games

The classical game theory so far considered considers the rather unrealistic case when the players have complete knowledge of all information relevant to the game. Relevant means not only that the players know strategy spaces and payoff functions of all players, but also that they know that the other players know all information, and know that other players know that players know . . . a regression *ad infinitum*. It is quite clear however that this is not the case in many realistic situations: one celebrated example in game theory is the case of the auction, for instance for an indivisible good: each player knows the valued he assigns to the good, but usually does not know the value assigned to the good by other players. Thus there is the need of developing a more sophisticated theory, in order to handle these more realistic situations. Here we briefly review Harsanyi's model of Bayesian game.

According to this model, each player can be of several types, and a type describes the player's belief on *the state of nature*, by this meaning *the data of the game* and all beliefs of any order. Let us put the idea in a formal definition. Let I be a finite set and $T_i : i \in I$ be finite sets. Set $T = \times_{i \in I} T_i$ and let S be a set.

Definition 7.0.1 *A (Harsanyi) game of incomplete information is:*

$$(I, S, T, Y \subset T \times S, p \in \Delta(Y)),$$

where for a finite set Y the set $\Delta(Y)$ represents the set of probability distributions on Y.

The meaning is the following:

- I represents the set of the players;
- S is the set of the states of nature;
- T_i is the set of the types of player i (we shall denote by T the set of all type profiles: $T = \times T_i$);
- Y represents the set of *states of the world*;
- p is the *common prior*.

A state of nature is a full description of a game; if the game is in strategic form, a state of nature s_k is of the form

$$s_k = (I, A_i, u_i^k : \times A_i \to \mathbb{R})),$$

where, as usual, A_i represents the set of the (pure) strategies of the player i and u_i his utility function. A state of the world y is a state of nature, and a list of types of the players: $y = (s, t_1, \dots, t_n)$. Observe that we assume the players and their actions independent from the state of nature. Only the utility functions vary in different states of nature.

Let us see some examples, in order to better understand the definition. We consider, to simplify things, only cases when the players are two. Generalization to more players is straightforward.

Example 7.0.1 Suppose we are given two 2×2 matrices M_1, M_2:

$$M_1 = \begin{pmatrix} (1,0) & (0,1) \\ (0,1) & (1,0) \end{pmatrix}, M_2 = \begin{pmatrix} (0,1) & (1,0) \\ (1,0) & (0,1) \end{pmatrix}$$

and suppose also that the first matrix is chosen by the nature with probability $p \in [0,1]$, known to both. Suppose moreover the outcome of the chance move is told only to player One. Let us see how to fit this situation in the Harsanyi model described above. The set S possible states of nature is $S = \{M_1, M_2\}$. It is natural to think that One is of two possible types: type $t_1 = I_1$ when knows that the matrix M_1 is selected, type $t_2 = I_2$ when the matrix M_2 is selected. Instead, the second player is of only one type $t_1 = II$ (Player One knows the utility functions of the second player when the game is played). Thus:

$$S \times T = ((M_1, I_1, II), (M_1, I_2, II), (M_2, I_1, II), (M_2, I_2, II)) .$$

However not all of elements of $S \times T$ make sense, thus

$$Y = ((M_1, I_1, II) := y_1, (M_2, I_2, II) := y_2) .$$

The prior on Y is $(p, 1-p)$. Observe that all states of nature correspond to a game, with actions and payoffs specified for all players. In particular, observe that the utilities of the players are determined by the specification of a state of the world and of an action for each player. Thus for instance the utility of the second player in state y_1 with actions T(op) for the first and R(ight) for the second is 1.

Example 7.0.2 Suppose the states of the nature are the same as in the previous game, but this time the first player does not know whether the second player is informed or not on the outcome of the chance move. The games goes as follows. A matrix is chosen with probability p. If the matrix M_1 is selected, then with probability q the second player is told that the selected matrix is M_1, but player I does not know it. In the case the matrix M_2 is selected, player One knows that player Two is not informed. Here the set S of possible states of nature is $S = \{M_1, M_2\}$. Both players have two possible types: player I is type I_1 when the matrix M_1 is selected, type I_2 when the matrix M_2 is selected. Player II is II_1 when informed, II_2 otherwise. The following states of the world make sense:

$$Y = ((M_1, I_1, II_1) := y_1, (M_1, I_1, II_2) := y_2, (M_2, I_2, II_1) := y_3) .$$

The prior on Y is $(pq, p(1-q), 1-p)$.

Example 7.0.3 The players are of two types, and of course each one knows his type, but not the other's one. Suppose the type of the player determines the belief on the type of the other player. We can assume, for instance, that I_1 believes that player II is type II_1 with probability $\frac{1}{4}$, I_2 believes that player II is type II_2 with probability $\frac{4}{6}$, while II_1 believes that player I is type I_1 with probability $\frac{1}{5}$, II_2 believes that player I is type I_1 with probability $\frac{3}{5}$. Accordingly, 4 games are given, f.i. 4 bimatrices of the same dimensions but with different entries, representing the payoffs of the players in the different situations. Then Y in this case consists of four elements and the prior in this example is

	II_1	II_2
I_1	$\frac{1}{10}$	$\frac{3}{10}$
I_2	$\frac{4}{10}$	$\frac{2}{10}$

We can easily verify this. In fact, given the prior, the players will update their beliefs in Bayesian sense. F.i. player I_1, infers that player Two is of type II_1 with probability $\frac{\frac{1}{10}}{\frac{1}{10}+\frac{3}{10}} = \frac{1}{4}$ and with routine calculations we get:

	II_1	II_2
I_1	$\frac{1}{4}$	$\frac{3}{4}$
I_2	$\frac{4}{6}$	$\frac{2}{6}$

	II_1	II_2
I_1	$\frac{1}{5}$	$\frac{3}{5}$
I_2	$\frac{4}{5}$	$\frac{2}{5}$

Example 7.0.4 Of course, a game with complete information, without chance moves, can be given the form of a Bayesian game: the state of nature will be just one element s (for instance a bimatrix). Thus Y reduces to a singleton, and the common prior assigns probability 1 to this element.

Example 7.0.5 Suppose to have a geme where nature makes some moves: to make things easier suppose this happens at the root of the game. Suppose moreover that nature selects among k alternatives $1, \dots, k$ with probabilities $p_1, \dots, p_k$. Then the states of nature can be represented by k matrices $M_1, \dots, M_k$, thus $S = \{M_1, \dots, M_k\}$. We assume $p_1, \dots, p_k$ is commonly known, and that the choice of nature is not told to the players. Thus $Y = (y_1, \dots, y_k)$ consists of k states of the world, and the players are of just one type, the same for all, which can be written as $p_1M_1 + \dots + p_kM_k$, to remind their probabilities on the states of the world.

Let us come back to Example 7.0.3. At the beginning, we suppose that the players have different types, and that each type of each player has beliefs on the other player's type. In the example, we were able to exhibit a common prior, but the question arises if this is always possible. The answer is (almost) obvious.

Example 7.0.6 Suppose to be in the same setting of Example 7.0.3, but this time suppose the players have the following beliefs on other player's types:

	II_1	II_2
I_1	$\frac{1}{2}$	$\frac{1}{2}$
I_2	1	0

$$\begin{array}{ccc} & II_1 & II_2 \\ I_1 & \frac{1}{3} & \frac{1}{4} \\ I_2 & \frac{2}{3} & \frac{3}{4} \end{array}.$$

Let us try to find a common prior. It should be of the form:

$$\begin{array}{ccc} & II_1 & II_2 \\ I_1 & x & y \\ I_2 & u & v \end{array},$$

($x, y, u, v \geq 0$ and $x + y + u + v = 1$). Then we must have, looking at I_2:

$$0 = \frac{v}{u + v},$$

so that $v = 0$. However, looking at II_2, we must have

$$\frac{3}{4} = \frac{v}{y + v},$$

and this is impossible.

We say that, in the above example, the beliefs of the players are *inconsistent*. How can we handle this case? Actually, there is a satisfactory theory in this case. Harsanyi motivated his assumption of a common prior by saying that when the players have inconsistent beliefs it is possible to make them adjust their probabilities in order to finally have a common prior. in any case, this is not a topic to consider here. What we need to do is to assume that the structure of the game is as in the Harsanyi definition, and so to define an equilibrium in this case.

Given that assumptions above, it is now time to define what is a strategy in this context, and an equilibrium for the game. Of course, we keep in mind the Nash idea. Actually, most of the job is already done. The concept of strategy requires to specify the behavior of a player *in any situation* she could be called to make a move. Thus, in our context a strategy for the player i is σ_i:

$$\sigma_i : Y \to A_i.^1$$

However, an important condition must be specified. *If the player i in two different states of the world is of the same type, then the strategy must specify the same choice.* The set of strategies for the player i is denoted by Σ_i while an element $(\sigma_1, \sigma_2) \in \Sigma = \times\Sigma_i$ is called a *strategy profile.*

We have all ingredients to define an equilibrium profile. But this is not complicated, actually most of the job has been done, since we have specified the idea of strategy. What we need is just to fix some notation. Thus, given one player, say player i, and a strategy profile (σ_1, σ_2), we shall write $u(y, \sigma_1(y), \sigma_2(y))$ to denote the utility of the player in the state of the world y and actions $\sigma_1(y)$ and $\sigma_2(y)$. Finally, the utility of player i when the strategy profile (σ_1, σ_2) is specified will be

[1] Observe that in the state of nature y a specific game is played. Thus $\sigma_i(y)$ is what is called a strategy for the player i in the game played in the state y. To avoid confusion, we use the term *action,* leaving the term strategy for the whole Bayesian game.

$$u_i(\sigma_1, \sigma_2) = \sum_y p(y) u_i(y, \sigma_1(y), \sigma_2(y)).$$

Needless to say, (σ_1^*, σ_2^*) is a Bayesian equilibrium strategy profile if σ_1^* (σ_2^*) is the best reply, for each type of player One (Two), to player Two (One):

$$u_I(\sigma_1^*, \sigma_2^*) \geq u_I(\sigma_1, \sigma_2^*),$$

$$u_{II}(\sigma_1^*, \sigma_2^*) \geq u_{II}(\sigma_1^*, \sigma_2).$$[2]

Let us go back to Example 7.0.2. Suppose the bimatrices M_1 and M_2 have two rows and two columns, and denote by L(eft) the choice of the first column and R(ight) that one of the second column. In a similar way denote by T(op) and B(ottom) the two possible choices of player One. The states of the world are three: y_1, y_2, y_3. Thus a strategy for player one must be a triple containing T and/or B, but the condition of consistency does not allow a strategy of the form $[T, B, T]$, for instance, since in the states y_1 and y_2 he is of the same type. Thus we shall write more simply (T, T), for instance, for a strategy of the first player. Analogously, only the following triples are strategies for the second player (L, L), (R, L), (L, R), (R, R). Finally, a strategy profile will be written, f.i. as $[(T, T), (L, L)]$.

To conclude, let us see in a simple case how to verify if the beliefs of the players are inconsistent. We consider the case of two players of two types each.
Suppose the players have the following beliefs on other player's types:

$$\begin{array}{ccc} & II_1 & II_2 \\ I_1 & p_1 & 1-p_1 \\ I_2 & p_2 & 1-p_2 \end{array}$$

$$\begin{array}{ccc} & II_1 & II_2 \\ I_1 & q_1 & q_2 \\ I_2 & 1-q_1 & 1-q_2 \end{array}.$$

Suppose the prior is given by the matrix

$$\begin{array}{ccc} & II_1 & II_2 \\ I_1 & x & y \\ I_2 & u & v \end{array},$$

$(x, y, u, v \geq 0$ and $x + y + u + v = 1)$. Suppose for the moment $x, y, u, v > 0$. We have that $\frac{x}{y} = \frac{p_1}{1-p_1}$, $\frac{u}{v} = \frac{p_2}{1-p_2}$ and thus:

[2] A slightly different notation will be used in the case the game considers the players having probabilities on the types of the other player, and the probabilities are coming from a given prior. In this case a matrix of probabilities $p(t_i, t_j)$ is given and the conditional probability of type i (of player I) over type j (of player II) is $p(t_j|t_i) = \frac{p(t_i,t_j)}{\sum_j p(t_i,t_j)}$. Thus, for player One of type t_i the (expected) utility according to the profile (σ_1, σ_2) is

$$u_1^i(\sigma_1, \sigma_2) = \sum_j u_1(\sigma_1(t_i), \sigma_2(t_j)) p(t_j|t_i).$$

Accordingly, the Bayesian equilibrium is the best replay, for player I, to the strategy of player II, and conversely.

$$\frac{xv}{yu} = \frac{p_1}{1-p_1}\frac{1-p_2}{p_2}.$$

In an analogous way we get:

$$\frac{xv}{yu} = \frac{q_1}{1-q_1}\frac{1-q_2}{q_2}.$$

It follows that:

$$\frac{p_1}{1-p_1}\frac{1-p_2}{p_2} = \frac{q_1}{1-q_1}\frac{1-q_2}{q_2}.$$

The case when one of the entries $x, y, u, v0$ is zero is left as an exercise.

Example 7.0.7 In the setting of Example 7.0.1, suppose the two matrices are:

$$\begin{pmatrix} (1,1) & (0,0) \\ (0,0) & (0,0) \end{pmatrix}, \begin{pmatrix} (0,0) & (0,0) \\ (0,0) & (2,2) \end{pmatrix}$$

and suppose also that the first matrix is chosen by the nature with probability $\frac{1}{2}$. Calling T, B and L, R the strategies of player One and Two respectively, we have then 8 possible profiles:

$$(T,T,L),(T,B,L),(B,T,L),(B,B,L),(T,T,R),(T,B,R),(B,T,R),(B,B,R).$$

Player One will not play B in M_1, if player Two plays left. This excludes the profiles $(B,T,L),(B,B,L)$. Similarly, she excludes $(T,T,R),(B,T,R)$. Now, let us write the utilities of player Two in all situations:

	L	R
TT	$\frac{1}{2}$	0
TB	$\frac{1}{2}$	1
BT	0	0
BB	0	1

From the table it is clear that Player Two does not accept (T, BL). All other situations give Bayesian equilibria.

7.1 Exercises

Exercise 7.1.1 Discuss the following game of incomplete information. Player II can be of two types, say II_1 or II_2. Player I attaches probabilities $(0.6, 0.4)$ to (II_1, II_2). Payments are as follows.

$$\begin{pmatrix} (1,2) & (0,1) \\ (0,4) & (1,3) \end{pmatrix}, \begin{pmatrix} (1,3) & (0,4) \\ (0,1) & (1,2) \end{pmatrix}$$

Solution We call T and B the choices of player I and L and R the choices of player II. L is strictly dominant for II_1 and R is strictly dominant for II_2. I evaluates his expected payoffs:

- playing T: $0.6(1) + 0.4(0) = 0.6$;
- playing B: $0.6(0) + 0.4(1) = 0.4$.

Then I plays T, while II_1 plays L and II_2 plays R, $[T,(L,R)]$ is Bayesian equilibrium.

Exercise 7.1.2 The Nature chooses one of the two following bimatrices with probability $\frac{1}{2}$. Player I can observe the result, while player II cannot. Show that $[(T,B),R]$ is a Bayesian equilibrium strategy profile:

$$\begin{array}{c} \\ T \\ B \end{array}\begin{array}{c} L \quad R \\ \begin{pmatrix} (1,1) & (0,0) \\ (0,0) & (0,0) \end{pmatrix} \end{array}, \quad \begin{array}{c} \\ T \\ B \end{array}\begin{array}{c} L \quad R \\ \begin{pmatrix} (0,0) & (0,0) \\ (0,0) & (2,2) \end{pmatrix} \end{array}$$

Solution We have 8 possible profiles $[(T,T),L]$, $[(T,B),L]$, $[(B,T),L]$, $[(B,B),L]$, $[(T,T),R]$, $[(T,B),R]$, $[(B,T),R]$ and $[(B,B),R]$. By domination we see that the only possible equilibria are $[(T,T),L]$, $[(T,B),L]$, $[(T,B),R]$ and $[(B,B),R]$), but it is obviuos that II cannot accept $[(T,B),L]$. All the others are Bayesian equilibria. For example, to prove $[(T,B),R]$ is an equilibrium, we verify with the definition that

$$u_I^1([(T,B),R]) \geq u_I^1([\sigma_I, R])$$

for each σ_I strategy of the first player, as $0 \geq 0$,

$$u_I^2([(T,B),R]) \geq u_I^2([\sigma_I, R])$$

for each σ_I strategy of the first player, as $2 \geq 0$, and

$$u_{II}([(T,B),R]) \geq u_{II}([(T,B),\sigma_{II}])$$

with $\sigma_{II} = L$, as $1 \geq \frac{1}{2}$.

Exercise 7.1.3 Consider an auction where a good is offered, the participants are two, the good is assigned to the person offering more to the offered price (first price auction). The type of each player is given by his evaluation of the good and it is uniformly distributed on $[0,1]$. No offer can be negative. Suppose that the utilities of the players are their evaluation minus the offer in the case they get the good, 0 otherwise. Show that, if the evaluation is t, the best offer is $\frac{t}{2}$.

Solution It is enough to show that

$$\forall t_i \in [0,1] \qquad \sigma_i(t_i) = \frac{t_i}{2}$$

is an equilibrium. Suppose player One evaluates t_i the good and player Two follows the above strategy. Suppose the offer of player One is x. Then her type t utility, given type θ of the other player, is

$$u(x,t) = (t-x)P(\theta/2 < x) = (t-x)2x.$$

It follows that utility is maximized by $x = \frac{t}{2}$.

Exercise 7.1.4 Find necessary and sufficient conditions, in the case of two players of two types each, under which the beliefs of the players are consistent with a common prior, when one of the players assigns probability zero to one type of the other player.

Solution Without loss of generality, we can assume that I_2 assigns probability zero to type II_2. It is immediate that player II_2 assigns probability zero to I_2. Thus, we need to analyze the following situation:

$$\begin{array}{ccc} & II_1 & II_2 \\ I_1 & p & 1-p \\ I_2 & 1 & 0 \end{array},$$

$$\begin{array}{ccc} & II_1 & II_2 \\ I_1 & q & 1 \\ I_2 & 1-q & 0 \end{array}.$$

If $p = 0$, then it easily follows that $q = 0$ and prior is made by:

$$\begin{array}{ccc} & II_1 & II_2 \\ I_1 & 0 & x \\ I_2 & 1-x & 0 \end{array},$$

with $0 < x < 1$. If $pq > 0$, with some calculation, it can be shown that the prior matrix has the following form:

$$\begin{array}{ccc} & II_1 & II_2 \\ I_1 & \frac{pq}{p+q-pq} & \frac{(1-p)q}{p+q-pq} \\ I_2 & \frac{p(1-q)}{p+q-pq} & 0 \end{array}.$$

Exercise 7.1.5 Consider an auction where a good is offered, the participants are two, the good is assigned to the person offering more to the offered price (first price auction). The type of each player is given by his evaluation of the good and it is uniformly distributed on $[0, 1]$. No offer can be negative. Suppose that the utilities of the players are their evaluation minus the offer in the case they get the good, 0 otherwise. Show that, if the evaluation is t, the best offer is $\frac{t}{2}$.

Solution It is enough to show that

$$\forall t_i \in [0, 1] \qquad \sigma_i(t_i) = \frac{t_i}{2}$$

is an equilibrium. Suppose player One evaluates t_i the good and player Two follows the above strategy. Suppose the offer of player One is x. Then her type t utility, given type θ of the other player, is

$$u(x, t) = (t - x)P(\theta/2 < x) = (t - x)2x.$$

It follows that utility is maximized by $x = \frac{t}{2}$.

Exercise 7.1.6 5 voters must elect three candidates, whose political position is, respectively, (L)eft, (C)enter, (R)ight. Suppose the voters can all be of the following four types (ordering their preferences): L-C-R, C-L-R, C-R-L, R-C-L. Suppose also that probability of each type, for each voter is uniformly distributed.

1. Prove that for each type voting her preferred candidate is *not* a weakly dominant strategy.
2. Prove that there is a Bayesian equilibrium such that for no type and for no player the vote is C.

Exercise 7.1.7 Write the normal form and find the pure Bayesian equilibria of the following game.

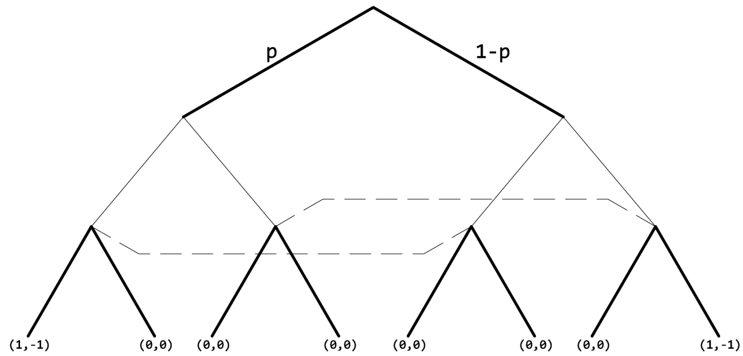

Fig. 7.1. Exercise 7.1.7

8

The bargaining problem

In the next two chapters, we shall consider two very nice applications of Game theoretical techniques. We shall present a pair of simplified and naive, yet interesting and meaningful, models aimed at answering some questions in common interaction problems. The first one deals with the model of bargaining, that could be described as the problem of finding the "right" way to distribute a given amount of good between two people. The second one instead is the problem of matching two groups of agents each of them having preferences on the agents of the other group. In this chapter we start to see the bargaining problem.
There are at least two types of possible approaches to the bargaining problem: for instance, we can try to model the typical situation of two people making alternating offers. Here instead we use another approach. It is supposed to know the utility of the two agents on all possible results of the distribution (think for simplicity, to the division of a given amount of money between them, but the model could apply to more general situations as well) and a solution is proposed on the basis of some reasonable properties the solution itself should fulfill. The prototype of these results is due to John F. Nash, Jr., which published his results in two papers written when he was student at Princeton, at the beginning of the fifties of the last century. Here we present it, together with one successive variant.

The very first point is to decide the type of mathematical objects which could represent a bargaining problem. According to Nash, a bargaining problem can be modelled as a pair (C, d), where $C \subset \mathbb{R}^2$ is bounded closed and convex set and $d \in C$ is a given point. The meaning is the following. Every vector $x = (x_1, x_2) \in C$ represents a distribution of utilities between the two players: x_1 is the utility relative to player one, x_2 that one relative to player two. d instead represents the utilities of the two players if they are unable to reach an agreement. Thus C represents the set of all pairs of available utilities the agents can get from the bargaining process. Let us see some words about the mathematical assumptions on C. Closedness and boundedness of C are very natural assumptions, the first one simply means that a pair of utilities arbitrarily close to available utilities is available, the second one that utilities arbitrarily big (in both positive and negative sense) are not admitted. Convexity instead requires a more sophisticated explanation, but can be accepted. It is also natural to assume the existence on C of a vector z such that $z >> d$ (this means $z_1 > d_1$, $z_2 > d_2$); otherwise, there is no reason for the players to bargain.
Denote by $\mathcal{C}$ the set of the bargaining problems, as described above.

Definition 8.0.1 *A* solution *of the bargaining problem is a function*

$$f : \mathcal{C} \to \mathbb{R}^2,$$

such that $f[(C, d)] \in C$*, for all* $(C, d) \in \mathcal{C}$.

It is quite clear that the above definition is useful, since it establishes the mathematical nature of the solution of the problem, but so far it says nothing about how to distribute utilities between the two bargainers, *which is the goal of the model.*

Thus the first question to be addressed is to list a number of properties that the function f should have. Nash proposed the following ones:

1. Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is the following transformation of the plane: $L(x_1, x_2) = (ax_1 + c, bx_2 + e)$, with $a, b > 0$ and $c, e \in \mathbb{R}$. Then
$$f[L(C), L(d)] = L[f(C, d)];$$
(*invariance with respect to admissible transformation of utility functions*);
2. Suppose $S : \mathbb{R}^2 \to \mathbb{R}^2$ is the following transformation of the plane: $S(x_1, x_2) = (x_2, x_1)$. Suppose moreover a game (C, d) fulfills $(S(C), S(d) = (C, d)$. Then
$$f(C, d) = S[f(C, d)];$$
(*symmetry*)[1]
3. Given the two problems (A, x) and (C, x) (observe, same disagreement point), if $A \supset C$, and if $f[(A, x)] \in C$, then $f[(C, x)] = f[(A, x)]$ (*independence from irrelevant alternatives*: for short IIA)
4. Given (C, x), if $y \in C$ and there is $u \in C$ such that $u >> y$, then $f[(C, x)] \neq y$ (*efficiency*).

A brief explanation of the properties required by Nash. The first one is a change of scale in the utilities. From the mathematical point of view, it says that if we change origin and measure units on the two axes in the planes, the solution changes accordingly (i.e. it does *not* depend from the way we represent the bargaining problem in the plane). A little more delicate is the economical interpretation of the assumption. What can be said is that if we assume that utilities can be represented in monetary terms, then the solution does not depend from the initial amounts of utility of the players, and from the currency we use to express these utilities (the numbers a, b are conversion factors between different utilities). Thus the result does not change if we express utilities in dollars rather than euros or Swiss francs. The second assumption can be interpreted in two different ways. First of all, observe that the game under consideration provides symmetric situation to the players, since a utility (x_1, x_2) is available if and only if the utility (x_2, x_1) is available as well. In one sense, it implies that the solution does not distinguish between two equal players: it can be called *anonimity*, and it seems to be very natural since no player should be in a advantageous position.[2] In another sense, we are assuming that their ability to negotiate is the same. Even this assumption is reasonable, under the rationality umbrella covering the whole

[1] A problem (C, d) fulfilling $(S(C), S(d) = (C, d)$ is called a *symmetric problem.*

[2] Assuming that they have the same available options.

theory. IIA instead in some sense means that if adding some alternatives to the set C does not bring the solution outside C, then the solution remains the same if we do *not* include those alternatives in C: though the mathematical formulation of this property is so natural to induce to believe that this assumption is acceptable in the model, actually this is probably the most questionable assumption. On the contrary, the last one is mandatory. It makes non sense to propose an inefficient distribution of utilities.

Here is the Nash theorem:

Theorem 8.0.1 *There is one and only one f satisfying the above properties. Precisely, if $(C, x) \in \mathcal{C}$, $f(C, x)$ is the point maximizing on $C \cap \{(u, v) : u \geq x_1, v \geq x_2\}$ the function $g(u, v) = (u - x_1)(v - x_2)$.*

In other words, the players must maximize the product of their utilities.

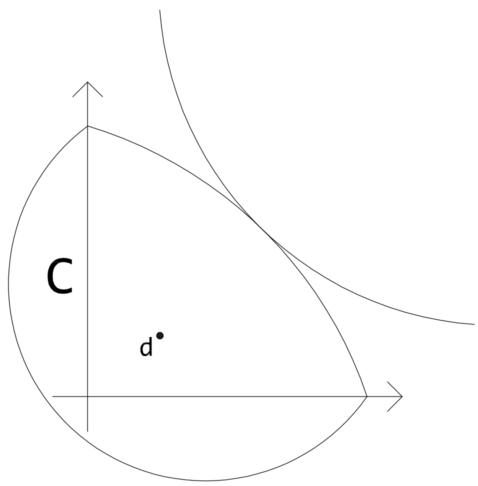

Fig. 8.1. The Nash solution of the bargaining problem

Proof. Outline. f is well defined: the point maximizing g on C exists, since g is a continuous function and the domain C is closed convex bounded). Uniqueness of the maximum point can be seen by remarking that the level curves of the functions g are hyperbolas. At the maximum point the level must be leave the set C below the curve. But this implies, draw a picture, that the point must be unique. This in turn implies that the solution is well defined, in the sense that it provides a unique point as a result, which implies that at least a solution is proposed. The verification that C satisfies the other properties in a bit boring but not difficult: we omit it. We only observe that IIA is trivial, that efficiency depends from the above considerations and is easy to show that f satisfies the above list of properties. uniqueness implies that the solution must belong to the the line $y = x$ in symmetric problems.[3] Smarter is the proof of uniqueness. Call h another solution. First of all, observe that properties 2 and 4 imply $h = f$ on the subclass of the symmetric games. Then take a general problem (C, x) and, by means of a transformation as in property 1, send x to the origin and the point $f(C, x)$ to $(1, 1)$. Observe that the set W obtained in this way is contained in the set $A = \{(u, v) : u, v \geq 0, \ u + v \leq 2\}$. Then $(A, 0)$ is a symmetric game, so that $f(A, 0) = h(A, 0) = (1, 1)$. The independence of irrelevant alternatives provides $h(W, 0) = (1, 1) = f(W, 0)$. Now via the first property go back to the original bargaining situation, and conclude from this.

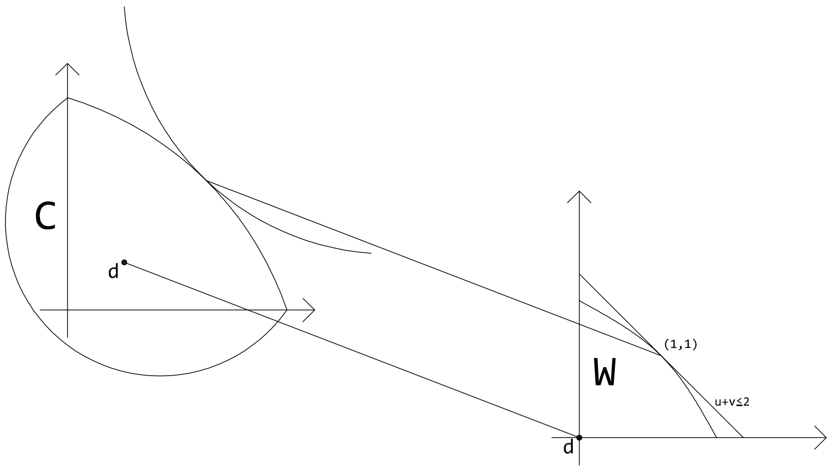

Fig. 8.2. How to prove uniqueness of Nash solution

Alternative models can be proposed by substituting one or more axioms in the Nash list, with some other desirable property. One rather popular alternative solution substitutes the IIA property with the following one, called the property of *individual monotonicity*: given a bargaining problem (C, d), let the *utopia* point U be defined as

$$U = (U_1, U_2),$$

where $U_i = \max u_i$ on $C \cap \{(u, v) : u \geq d_1, v \geq d_2\}$ (in other words U_i is the maximum the player i can get in the bargaining process). For $x \leq U_1$, let moreover

$$g_C(x) = \begin{cases} y & \text{if } (x, y) + \mathbb{R}^2_+ \cap C = (x, y) \\ U_2 & \text{else} \end{cases} .$$

[3] If the maximum is attained at a point $(x, 0$, then it is also attained at the point (y, x). Since the point us unique, then $x = y$.

Observe that g_C associates to the amount of utility *at least* x of the first player the maximum available utility y of the player two. The monotonicity property for player two then reads as follows:

Let (C, d), $(\hat{C}, d)$ be two problems such that $U_1[(C, d)] = U_1[(\hat{C}, d)]$ and $g_C \leq g_{\hat{C}}$. Then $f_2[(\hat{C}, d)] \geq f_2[(C, d)]$.

The *Kalai-Smorodinski* solution is then found by taking the segment joining the disagreement and utopia points and by considering the (unique) point of this line segment lying on the boundary of C.

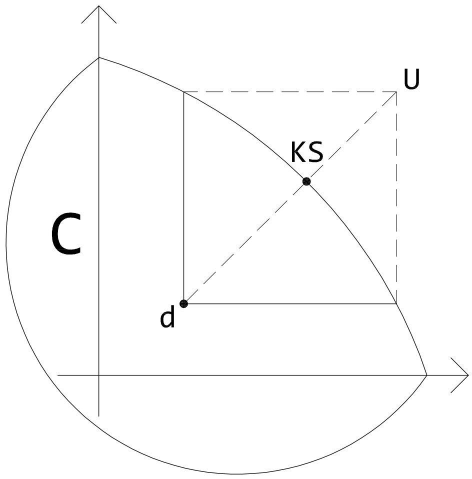

Fig. 8.3. The solution of Kalai-Smorodinski

The property of individual monotonicity can be explained in the following way: first of all, observe that, due to efficiency, which is always reasonable to require, a problem (C, d) is equivalent to the problem $(\bar{C}, d)$, where $\bar{C} = \{(x, y) : \ x \leq U_1, y \leq g_C(x)\}$. Then the property essentially requires that if, by enlarging the set of choices for player two, player one does not change his maximal available utility, then player two must get no less than before.

Here is the theorem:

Theorem 8.0.2 *There is one and only one* f *satisfying the properties of invariance with respect to admissible transformation of utility functions, symmetry, efficiency and monotonicity for both players. Precisely, it is the Kalai-Smorodinski solution.*

To conclude this chapter, let us study the situation of dividing 100 between two players under these assumptions: let h be an increasing, concave and twice differentiable function such that $h(0) = 0$. We suppose the players have utilities u_1 and u_2 respectively,with u_1 u_2 as h above, and characterize the solution; moreover, we see what happens when one of the two players changes his utility function in an appropriate way.

First of all, observe that when the players have the same utility function u, the set C of admissible outcomes is symmetric. Thus both agents receive 50, with utility $u(50)$. Now, assume they have different utilities, say u_1, u_2 respectively. Call z the amount received by the first player according to the Nash solution. Then z must maximize

$$g(x) = u_1(x)u_2(100 - x).$$

Since g is nonnegative and $g(0) = g(100) = 0$, its maximizing point must be in the interior of the interval $[0, 100]$. Thus it must be $g'(z) = 0$. This leads to the equation:

$$\frac{u_1'(z)}{u_1(z)} = \frac{u_2'(100 - z)}{u_2(100 - z)}.$$

It is easy to see that $z \mapsto \frac{u_1'(z)}{u_1(z)}$ is a strictly decreasing function, and thus the two curves $\frac{u_1'(z)}{u_1(z)}$ and $\frac{u_2'(100-z)}{u_2(100-z)}$ must intersect at a (unique) point. Thus the equation above characterizes the solution. Now, suppose the second player changes his utility function from u_2 to $h \circ u_2$. The equation determining the solution becomes:

$$\frac{u_1'(z)}{u_1(z)} = \frac{h'(u_2(100 - z))u_2'(100 - z)}{h(u_2(100 - z))}.$$

By the assumptions made on h, one easily sees that $h'(t) \leq \frac{h(t)}{t}$, and this finally implies that the amount z given to the first player is augmented with respect to the first case.

Economists say that the second bargainer is in the second case more *risk averse*. Thus the conclusion is that becoming more risk averse results in having less, at least according to the Nash bargaining scheme.

8.1 Exercises

Exercise 8.1.1 Show that the KS solution is the unique one fulfilling the monotonicity property and all properties of the Nash solution but IIA.

Solution We consider a normalized problem with bargaining set S_1 as in Figure 1, with $d = (0, 0)$ and $U = (1, 1)$. We consider $S_2 = co\{(0, 0), (1, 0), KS(S_1), (0, 1)\}$, where $KS(S_1)$ is the KS solution for (S_1, d). The bargaining problem (S_2, d) is symmetric, then for a given solution f fulfilling the same properties of KS, we have $f(S_2) = KS(S_2) = KS(S_1)$. Because of the individual monotonicity, $f_2(S_1) \geq f_2(S_2)$ and $f_1(S_1) \geq f_1(S_2)$. But this implies $f(S_1) = f(S_2)$ (it does not exist any point in S_1 different from $(f_1(S_2), f_2(S_2))$ which satisfies this property).

Exercise 8.1.2 Two guys, one rich and one poor, must decide how to share 500 Euros between them. If they do not agree, they will get nothing. The rich guy, when receiving the amount l of money, will get a satisfaction $u_1(l) = cl$, where $c > 0$. The utility function of the poor guy is instead $u_2(l) = \ln\left(1 + \frac{l}{100}\right)$. Find what Nash proposes to the players.

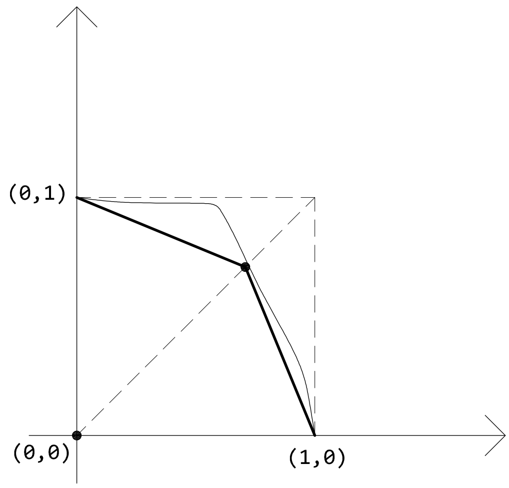

Fig. 8.4. Exercise 8.1.1

Solution When player 1 gets l, player 2 gets $500 - l$, then we have to maximize

$$l \ln \left(\frac{600 - l}{100} \right),$$

by searching for the critical point in the interval $(0, 500)$. The derivative at the middle point of the interval is positive, and this implies that the maximum will be on the right of the middle point and the first player will take more than one half of the amount. Approximatively, the rich will get 310 Euros and the poor 190.

Exercise 8.1.3 Two players I and II have to divide an amount of 100 euros. Their utilities are $u(x) = x$ and $v(x) = \sqrt{x}$ respectively. Make a picture of the bargaining set and find the Nash solution and the KS solution of the problem when the disagreement point is $d = (0, 0)$.

Solution We have that the bargaining set is given by the relation $u + v^2 \leq 100$. So finding the maximum of uv on this set is equivalent to finding the maximum of $v(100 - v^2)$, which is given by $v = \sqrt{\frac{100}{3}}$. The Nash solution is indeed $\left(\frac{200}{3}, \sqrt{\frac{100}{3}}\right)$. KS will be given by the intersection of $u = 100 - v^2$ and $u = 10v$ (the utopia point is $(100, 10)$), then $(50(\sqrt{5} - 1), 5(\sqrt{5} - 1))$.

Exercise 8.1.4 Given the bargaining problem (C, d), where $d = (2, 1)$ and $C = \{(u, v) : (u - 2)^2 + 4(v - 1)^2 \leq 8\}$

1. make a picture of the bargaining set;
2. find the Nash solution;
3. find the Kalai-Smorodinski solution.

Solution We have to maximize $(u - 2)(v - 1)$ on

$$\begin{cases} (u-2)^2 + 4(v-1)^2 = 8 \\ u \geq 2, v \geq 1 \end{cases}$$

if $t = v - 1$ this is equivalent to the problem of maximizing

$$t^2(2 - t^2), \qquad 0 \leq t \leq \sqrt{2}.$$

The solution is $t = 1$, i.e. $v = 2$ e $u = 4$. The Nash solution is indeed given by $(4, 2)$. This point is also the KS solution as it stays on the line connecting $d = (2, 1)$ and $U = (2 + 2\sqrt{2}, 1 + \sqrt{2})$.

Exercise 8.1.5 Characterize geometrically the Nash solution when the bargaining set is of the form

$$C = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ ax + by - c \leq 0\} \quad d = (0, 0),$$

where a, b, c are positive reals. Generalize when the efficient frontier is made by two segments. In this case characterize, given a point x of the efficient frontier, all points d of the bargaining set having x as Nash solution. And what about the KS solution in this case?

Solution We find the maximum on $ax+by-c = 0$. We have to maximize $x(-\frac{a}{b}x+\frac{c}{b})$. The Nash Solution is given by $\left(\frac{c}{2a}, \frac{c}{2b}\right)$. We notice this is the middle point of the segment (Figure 2, we could have solved the problem by a change of coordinates, Property IAT).
This is the Nash solution for every point of the line connecting $(0, 0)$ with the Nash solution.
When we have that the efficient frontier is made by two segments, if the middle point of both is outside the bargaining set, the Nash solution is given by the intesection between the two straight lines and it is Nash solution for every disagreement point as in Figure 3. If one of the two middle points is inside, this is the Nash solution and it remains solution for every disagreement point on the segment connecting it with $(0, 0)$, as it is shown in Figure 4.
This is not true for the KS solution.

Exercise 8.1.6 Given the bargaining problem (C, d) where $C = \{x_1^n + x_2^n \leq 1, x_1 \geq 0, x_2 \geq 0\}$

1. find the Nash and KS solutions when $d = (0, 0)$;
2. find the KS solution when $d = (x, 0), 0 \leq x \leq 1$;
3. show that the Nash and KS solutions with $d \in C$ cannot have both the coordinates rational.

Solution

1. The problem is symmetric, the Nash and the KS solution are the point $\left(\frac{1}{\sqrt[n]{2}}, \frac{1}{\sqrt[n]{2}}\right)$, i.e. the symmetric point on the boundary.
2. Assuming $d = (x, 0)$, we find the utopia point $U = (1, \sqrt[n]{1 - x^n})$. The intersection between the line connecting d and U and the boundary of the bargaining set is the KS solution.
3. Because of the Fermat Theorem they cannot have both the coordinates rational.

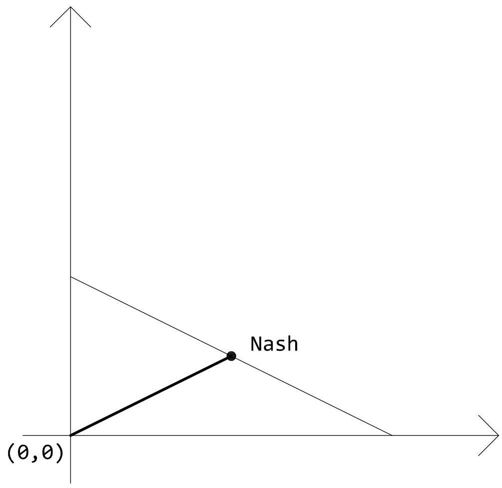

Fig. 8.5. Exercise 8.1.5

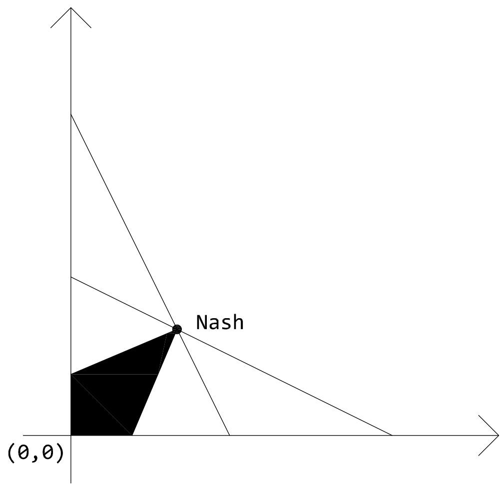

Fig. 8.6. Exercise 8.1.5

Exercise 8.1.7 Find an example showing that the Nash solution does not fulfill the individual monotonicity property.

Solution With $d = (0, 0)$ consider

$$C = \mathrm{co}\{(0, 0), (30, 0), (25, 15), (0, 20)\} \; \hat{C} = \mathrm{co}\{(0, 0), (30, 0), (20, 16), (0, 20)\}.$$

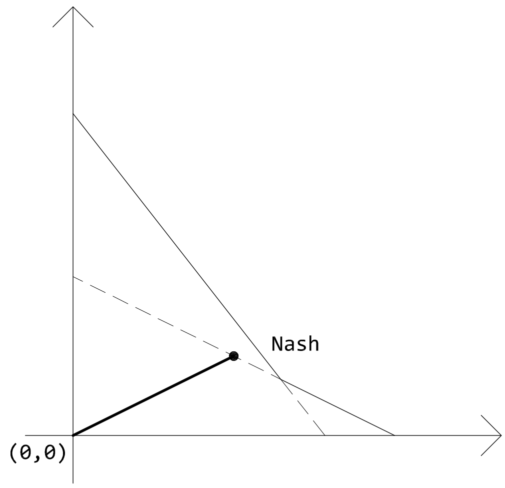

Fig. 8.7. Exercise 8.1.5

In the first problem the Nash solution is $(25, 15)$, in the second one $(20, 16)$. As $15 < 16$, this shows that the Nash solution does not fulfill the individual monotonicity property.

Exercise 8.1.8 A producer P produces one good, sold by a seller S. P works $a \leq 36$ hours per week and her salary is w per hour, then her utility is $u_P(a, w) = aw$ (her salary). S can sell one good at the unitary price $36 - a$, then her utility is $u_S(a, w) = (36 - a)a - aw$ (his profit). Find the Nash and KS solutions supposing in both cases $d = (0, 0)$.

Solution In order to maximize her utility (profit), S has to sell the quantity $\bar{a} = \frac{36-w}{2}$, with a gain $u = u_S(w) = \frac{(36-w)^2}{4}$ and with constraint $u \in [0, \frac{36^2}{4}]$. The utility of P in such a case is $v = u_P(w) = \frac{36-w}{2}w$. By eliminating the variable w we get $v = 36\sqrt{u} - 2u$. Observe that the efficient frontier of the set is when $u \in [81, 324]$. The Nash solution of the problem is $(182, 302)$. The salary will be indeed $w = 9$.
The utopia point is $U = (324, 162)$. The straight line connecting U and d is $v = \frac{1}{2}$. The KS solution will be given by the intersection with $v = 36\sqrt{u} - 2u$ and it will be $(207, 104)$. In this case the salary will be $w = 7$.

Exercise 8.1.9 Two players bargain to divide an amount of 1. The first one has utility $u(x) = \frac{1}{2}x$, while the second one $v(x) = 4 \sin x$. Who takes more, according to Nash?

Solution The utility of the first player is linear in x. This shows that the second player is more risk averse than the first one and thus the second gets less.

Exercise 8.1.10 Given the bargaining problem $d = (0, 0)$ and $C = \{(x_1 + 2x_2 \leq 8, 3x_1 + x_2 \leq 12, x_1, x_2 \geq 0\}$

1. draw the bargaining set;

2. find the Nash solution;
3. find the KS solution.

Solution

1. The two segments obtained intersecting the two straight lines with the axis have both middle point outside the bargaining set. It follows that the Nash solution is given by the intersection between them: $\left(\frac{16}{5}, \frac{12}{5}\right)$;
2. the utopia point is $U = (4, 4)$. The line connecting d and U is $x_2 = x_1$ and the point on the boundary is $\left(\frac{8}{3}, \frac{8}{3}\right)$.

Exercise 8.1.11 Given the convex hull of points $(0, 0)$, $(3, 0)$, $(0, 3)$, $(2, 2)$

1. find the Nash solution and the Kalai-Smorodinski solution when $d = (0, 0)$;
2. find every value of d for which $(2, 2)$ is the Nash solution.

Solution

1. The game is symmetric, then $(2, 2)$, the symmetric point on the boundary, is the Nash and the KS solution;
2. according to the Exercise 2.8.4, $(2, 2)$ is the Nash solution for every point in $co\{(0, 0), (1, 0), (2, 2), (0, 1)\}$.

Exercise 8.1.12 Two players bargain to divide an amount of b euros. The first one has utility $u(x) = \ln(1 + \frac{x}{b})$, while the second one $v(x) = \sqrt{x}$.

- Find the utopia point when b varies over the positive reals;
- Find all b such that the second player gets less than the first one, according to Nash.

9

The marriage problem

In this chapter we see one more example how game theory faces problems with multiple aspects, from finding the right idea of "solving" a social interaction problem, to providing an existence result for an efficient outcome and even more, to *ranking* the solutions by showing which type of agents will be better off by following certain procedures. As usual, these models introducing new topics are very naif, but on the other side they are the right first step for more sophisticated theories.

There are two groups of players. All players of the first (second) group have preferences on the players of the second (first) group: typically, they could be men and women willing to get married. The problem is: no matter their preferences are, is there an "optimal" way to match them? Let us try to make more precise what we need to do. First of all, we have to *define* what an optimal way to match them is. Next, when we have a satisfactory concept of solution (outcome) of the game, the natural issue is whether the outcome is guaranteed no matter are the preferences of the two groups. Furthermore, are different outcomes possible? In this case, is it possible to say which one are better, with respect to the other ones, for a set of players? Finally, can we find an algorithm to find out solution(s) of a given game?

We shall see that for this problem we can give answers to all the previous questions. To introduce the setting, let us start by a simple example, by imagining that the groups are $M = \{A, B, C\}$ and $W = \{a, b, c\}$. Let us propose one possible matching situation: $\{(A, a), (B, b), (C, c)\}$. When is it not stable? For instance, if A likes better b than a, and b likes better A than B. Thus a situation can be defined stable if it does *not* happen something like above, since this guarantees that if one is willing to change his/her partner with another person this last one will not accept the proposal. Let us formalize what we did. There are two groups M an W,of agents, and each agent in M (W) has preferences on the set W (M). A *stable set* for the problem is a *set $\mathcal{S}$ of pairs with the following property:*

if $(a, B), (c, D)$ are pairs in $\mathcal{S}$, with $a, c \in M$, $B, D \in W$ and if a likes better D than B, then D likes better c than a.

We have a satisfactory answer to the first issue: a good solution concept has been defined.
Now the next problem is to establish existence of the solution. In order to do this, let us make the following assumptions:

1. the two groups have the same amount of players;

2. the preferences are strict (i.e. nobody is indifferent to two members of the other group);
3. everybody likes better to be paired to somebody than to be alone.

Then, let us suggest the following procedure:

- *Step* 1 (First day). Select a group, for instance men, and tell them to go to visit their most preferred woman, according to their preference profile. Then look at what happens: either a woman has two or more different men, and in that case she selects her preferred one and kicks out the other ones, or no woman has a choice (and thus all pairs are formed), and the process ends. In the first case, go to the second step;
- *Step* 2 (Second day). The men rejected at the first step visit their second choice. The women having either more than a man visiting her or a man visiting her and another one selected at the first step, selects her more preferred man (observe, a man accepted at the first step can be refused at the second step). Now, if no woman has a choice, the process ends, otherwise go the next step;
- *Step* 3 . . . (Following days) In case there is a man rejected, repeat the procedure of the second day.

Now, at least two questions are in order: does this process ends? And in case it ends, do the obtained pairs form a solution of the problem?
The answer is not difficult, to both questions. Certainly, the process will end: a very rough upper bound of the days spent in the process is given by supposing that every man visits every woman. Is the final situation a stable one? The answer is positive! Suppose Bill likes better Caroline than Sarah, matched to him with our system. This means that he already visited Caroline, who then choose another man. This in turn means that Caroline will not accept any proposal by Bill. Of course, Bill will not make any proposal to a woman he likes less than Sarah. This means that the procedure provides a stable set.
Observe also that the proof is *constructive*, in the sense that we found an algorithm to single out a solution. Furthermore, it is clear that the solution in general will not be unique, since by interchanging the role of the two groups in general two different stable situations will be implemented. We shall call $\mathcal{M}$, $\mathcal{W}$ respectively the two stable sets obtained by the procedure with men, respectively, women visiting.
Probably the most interesting result is however the fact that we are able to classify all possible stable outcomes, for, it can be shown that the above procedure gives the best possible outcome to the men, than in any other stable situation, in the sense that we now specify.
Given a man, say X, let $A(X)$ be the set of women paired to him in some stable set ($A(X) = \{w :$ there exists $\mathcal{S}$ stable set with $(X, w) \in \mathcal{S}\}$).

Theorem 9.0.1 *$\mathcal{M}$ provides to every man X his most preferred woman among those in $A(X)$.*

Proof. Outline (complete by homework). Prove that no man can be rejected the first day by an available woman. Then prove that if no man is rejected the days $1, \dots, k-1$, then no man is rejected the $k-$th day.

The next is another interesting result. It characterizes uniqueness of the stable set.

Proposition 9.0.1 *Prove that the stable set is unique if and only if $\mathcal{M} = \mathcal{W}$.*

Proof. Homework.

Of course, there are many variants to the above model. For instance, we could suppose that the number of men and women is different, that the option to remain alone, rather than being paired with somebody is considered, and so on. Perhaps, more interestingly, it could be observed that usually preferences are private information, and thus for instance in the above procedure women, supposed to know the men's tastes, could be induced to lie in order to get a better mate, and thus study mechanisms in order to avoid this, and so on. Furthermore, other similar, but not exactly identical, situations can be considered, think for instance to the problem of assigning n twin bed rooms to $2n$ students.

A final remark: this example shows that a procedure to arrive to an equilibrium could be considered "politically incorrect" (a person yesterday accepted could be rejected today). But this is the price to be paid in order to get an equilibrium (note, the best equilibrium for the players risking to be rejected): introducing constraints, like for instance politician love to do, could finally produce a bad, unstable situation.

At the following address you can find an applet calculating a stable set for at most 16 men and women with randomly generated preferences.
http://www.dcs.gla.ac.uk/research/algorithms/stable/EGSapplet/intro.html

9.1 Exercises

Exercise 9.1.1 Prove Proposition 9.0.1.

Solution Suppose there is another stable set, say $\mathcal{J}$. Then it exist a man X and women w_1, w_2 such that

$$(X, w_1) \in \mathcal{J} \quad \wedge \quad w_2 \in A(X) \quad \wedge \quad w_2 \succ_X w \text{ for all } w \in A(X).$$

But then $X \in A(w_2)$ and $A \succ_{w_2} B$ for all $B \in A(w_2)$ and this is against the stability of $\mathcal{J}$.

Exercise 9.1.2 Find a 3 men 3 women situation with preferences such that the stable sets are more than two.

Solution Let men be A, B, C respectively, women 1, 2, 3. Consider the following preference systems:

$$\begin{array}{lllll} 1 & \succ_A 3 & \succ_A & 2 \\ 2 & \succ_B 1 & \succ_B & 3 \\ 3 & \succ_C 2 & \succ_C & 1, \end{array}$$

$$\begin{array}{lllll} C & \succ_1 B & \succ_1 & A \\ A & \succ_2 C & \succ_2 & B \\ B & \succ_3 A & \succ_3 & C. \end{array}$$

$(1, A)$, $(2, B)$ and $(3, C)$ is the stable solution obtained when we suppose that men go to visit women, $(1, C)$, $(2, A)$ and $(3, B)$ is the stable solution obtained when we suppose that women go to visit men, $(1, B)$, $(2, C)$ and $(3, A)$ is a third stable solution when men chose their second best choice.

Exercise 9.1.3 4 boys have to divide two two bed rooms in an apartment. Is this the same type of problem as the marriage problem? If not, does a stable set always exist?

Solution Let A, B, C and D be the four guys, we consider the following preferences

$$B \succ_A C \succ_A D,$$

$$C \succ_B A \succ_B D,$$

$$A \succ_C B \succ_C D.$$

Preferences of D are irrelevant. Nobody wants to stay with him and every matching is unstable since the person matched to him can make a proposal that will be accepted. For example if the match is given by (A, B), (C, D), C may make a proposal to B, who will accept since he prefer C instead of A. This happens with every possible match we choose.

Exercise 9.1.4 Complete the proof of Theorem 9.0.1.

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