Extensive form games

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Contents of the week

- Extensive form games
- Backward induction
- Games for which nothing is known about equilibria
- Games for which something is known
- Games strongly solvable
- Strategies

Extensive form

Three politicians must vote if to raise or not their salaries. Vote is public and in sequence. Their first option is to have an increase in the salaries, but they would like to vote against.

Main features

1. The moves are in sequence
2. Every possible situation is known to the players, at any time they know the whole past history, and the possible developments

Games with perfect information

How can we represent them?

How can we solve them?

The tree

The tree

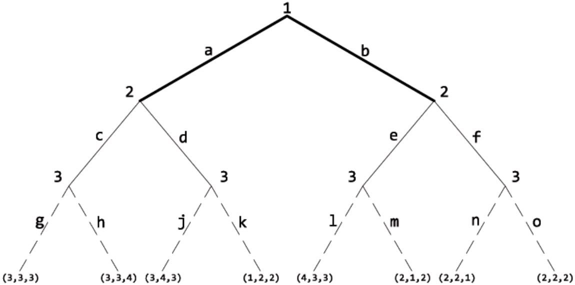
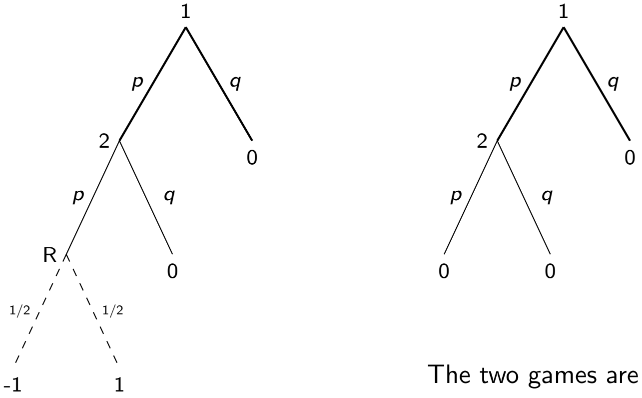


Figure: The game of the three politicians

A game with chance

The players must decide in sequence whether to play or not. If both decide to play, a coin is tossed, and the first one wins if the coin shows head, otherwise is the second to win.



The two games are equivalent.

Directed graphs

The tree of the game is a special type of oriented graph, where there are vertices and edges. Each vertex represents a possible situation of the game.To each vertex a label is associated, with the name of the player having to move at that situation. Edges represent the possible moves of the player. A child c of a vertex v is any vertex connected to v, which means a possible immediate situation of the game, after the situation described by the vertex v. A vertex with no children represents an outcome of the game, and is called a leaf. To complete the description of the game to each leaf the utility of the players at that outcome must be specified.

A path from a vertex v_1 to a vertex v_{k+1} is a finite sequence of vertices-edges $v_1, e_1, v_2, \dots, v_k, e_k, v_{k+1}$ such that $e_i \neq e_j$ if $i \neq j$ and $e_j = (v_j, v_{j+1})$. k is called the length of the path. A path from the root to a leaf represents a possible play of the game.

Define Length of the game as the length of the longest path in the game.

Backward induction

A game of length 1 is solvable since only one player takes a decision (so she maximizes her utility)

This is known to all players. Thus a game of length 2 is solvable since at any node such that the subtree has length 2 or less the player maximizes her utility, since she knows to which outcome bring all her possible moves

The process can be iterated and thus we can solve games of any finite length

This method is called the method of backward induction

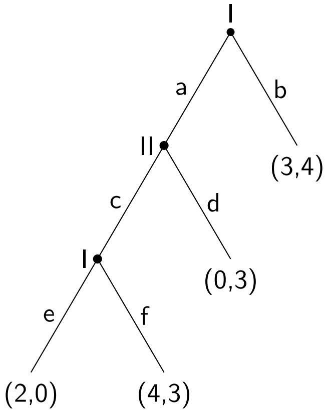
The first rationality theorem

Theorem

The rational outcomes of a finite, perfect information game are those given by the procedure of the backward induction.

Observe: this method relies on the fact that every vertex v of the game is the root of a new game, made by all followers of v in the initial game. This game is called a subgame of the original one

Multiple solutions



The outcomes obtained by backward induction are: $(4, 3)$ and $(3, 4)$. Thus uniqueness is not guaranteed.

Strategies

In Backward induction a move must be specified at any node. P_i is the set of the nodes where player i is called to make a move.

Definition

A pure strategy for player i is a function defined on the set P_i, associating to each node v in P_i a child w, or equivalently the edge (v, x).

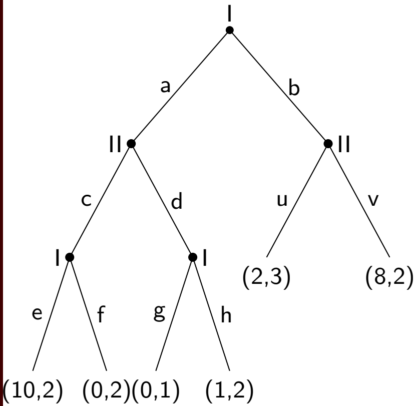
A mixed strategy is a probability distribution on the set of the pure strategies.

When a player has n pure strategies, the set of her mixed strategies is

$$\Sigma_n := \{p = (p_1, \dots, p_n) : p_i \geq 0 \sum p_i = 1\}$$

Σ_n is the fundamental simplex in n-dimensional euclidean space

Example



	cu	cv	du	dv
aeg	(10,2)	(10,2)	(0,1)	(0,1)
aeh	(10,2)	(10,2)	(1,2)	(1,2)
afg	(0,2)	(0,2)	(0,1)	(0,1)
afh	(0,2)	(0,2)	(1,2)	(1,2)
beg	(2,3)	(8,2)	(2,3)	(8,2)
beh	(2,3)	(8,2)	(2,3)	(8,2)
bfg	(2,3)	(8,2)	(2,3)	(8,2)
bfh	(2,3)	(8,2)	(2,3)	(8,2)

Observe: if $\{v_1, \dots, v_k\}$ is the set of vertices of one player and each v_j has n_j children the number of strategies of Player i is $n_1 \cdot n_2 \cdot \dots \cdot n_k$. This shows that the number of strategies even in short games is usually very high. Observe also that the above table has pairs repeated several times: different strategies can lead to the same outcomes.

The chess (Zermelo) theorem

Backward induction can in principle be applied to the chess game.

Theorem

In the game of chess one and only one of the following alternatives holds:

1. *The white has a way to win, no matter what the black does*
2. *The black has a way to win, no matter what the white does*
3. *The white has a way to force at least a draw, no matter what the black does, and the same holds for the black*

Note, it is impossible to say more than that.

Revisiting Zermelo

In terms of strategies here is Zermelo's theorem:

Theorem

In the chess game one of the following alternatives holds:

1. *the white has a winning strategy*
2. *the black has a winning strategy*
3. *both players have a strategy leading them at least to a tie*

Outcomes chess in strategic form 1

Here the white is the row player and in the entries the outcome of the game (W white wins, B black wins, T the outcome is a tie, is indicated)

	1	2	3	...	K
a	...	...	...	...	...
b	...	...	...	...	...
c	W	W	W	...	W
...	...	...	...	...	...
K	...	...	...	...	...

The White has a winning strategy

Outcomes chess in strategic form 2

Here the white is the row player and in the entries the outcome of the game (W white wins, B black wins, T the outcome is a tie, is indicated)

	1	2	3	...	K
a	...	...	B	...	...
b	...	...	B	...	...
c	...	...	B	...	...
...	...	...	...	...	...
K	...	...	B	...	...

The Black has a winning strategy

Outcomes chess in strategic form 3

Here the white is the row player and in the entries the outcome of the game (W white wins, B black wins, T the outcome is a tie, is indicated)

	1	2	3	...	K
a	...	...	B	...	...
b	...	...	T	...	...
c	T	W	T	...	W
...	...	...	...	...	...
...	...	...	T	...	...

The outcome of the game is a Tie

This is excluded

T	B	W
W	T	B
B	W	T

This is the case excluded by Zermelo

Extending Zermelo theorem

The Zermelo theorem applies to every finite game of perfect information where the possible result is either the victory of one player or a tie. Thus, if the tie is not allowed, the following Corollary holds

Corollary

Suppose to have a finite perfect information game with two players, and the outcomes are the victory of either player. Then one and only one of the following alternative holds:

1) *The first player has a way to win, no matter what the second one does*
2) *The second player has a way to win, no matter what the first does*

Different types of solutions

Does the chess theorem say anything useful?

Very weak solution

The game has a single rational outcome, inaccessible like for the Chess

Weak solution

The outcome of the game is known, but how to get it is not

Solution

It is possible to provide an algorithm to find a solution

Chomp

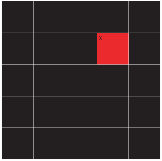


Figure: The first player removes the red square

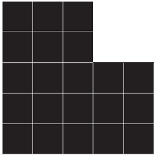


Figure: Now it is up to the second player to move

The player taking the most left-down square looses the game

Chomp

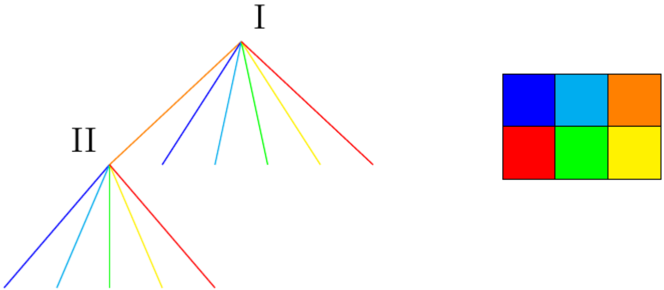


Figure: Edges are coloured according to the chosen square

Suppose the second wins. Then he has a winning action after the choice of the orange edge. Suppose the winning move for him is to choose the green edge. This means that the player starting at the node following the green edge will loose. But the tree starting at that node is exactly the same tree starting at the green edge relative to the first player. Thus the game starting at the node following the green edge going out from the root of the tree (move of the first player) has the starting player (i.e. the second player) loosing. Contradiction!

Finding the solution

Definition

An impartial combinatorial game is a game such that

1. *There are two players moving in alternate order*
2. *There is a finite number of positions in the game*
3. *Both players have the same rules to follow*
4. *The game ends when no moves are possible anymore*
5. *Chance is not present in the game*
6. *In the classical version the winner is the player leaving the other player with no available moves, in the misère version the opposite*

Examples of combinatorial games

1. k piles of cards. At her turn the player takes as many cards as she wants (at least one!) from one and only one pile
2. k piles of cards. At her turn the player takes as many cards as she wants (at least one!) from not more than $j < k$ piles
3. k cards in a row. At her turn the player takes either j_1 or $\dots$ or j_l objects

In the first two cases the positions are $(n_1, \dots, n_k)$ where n_i is a non negative integer for all i. In the last examples positions can be seen as all non negative integers smaller or equal to k.

The idea to solve these games

Partition the set of all possible positions into two sets:

1) *P*-positions
2) *N*-positions

Rules:

1. terminal positions are *P*-positions[1]
2. From a *P*-position only *N*-positions are available
3. From an *N*-position it is possible to go to a *P*-position

Player playing from an *N*-position wins!

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[1] Terminal means that the player does not have a possible move (classical version)

The Nim game

Nim game is defined as $(n_1, \ldots, n_k)$ where for all i n_i is a positive integer. A player at her turn has to take one (and only one) n_i and substitute it with $\hat{n}_i < n_i$. The winner is the player arriving to the position $(0, \ldots, 0)$.

Meaning: taking away cards form one pile. Goal: to clear the table.

A new operation on the non negative integers

Define an operation $\oplus$ on $\mathbb{N} = \{0, 1, 2, \dots, n, \dots\}$ in the following way: for $n_1, n_2 \in \mathbb{N}$

1. Write n_1, n_2 in binary form (notation for n in binary form $[n]_2$)
2. Write the sum $[n_1]_2 \oplus [n_2]_2$ in binary form where $\oplus$ is the (usual) sum but without carry
3. The result is the obtained number, written in binary form

For instance, since

$$7 = 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$$

then $[7]_2 = 111$.

$$9 = 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$

then $[9]_2 = 1001$.

An example

Example

The $\oplus$ operation applied to 1,2,4, and 1.

$$\begin{array}{rcccc} [1]_2 & = & 0 & 0 & 1 \\ [2]_2 & = & 0 & 1 & 0 \\ [4]_2 & = & 1 & 0 & 0 \\ [1]_2 & = & 0 & 0 & 1 \\ \hline \\ [6]_2 & = & 1 & 1 & 0 \end{array}$$

$$n_1 \oplus n_2 \oplus n_3 \oplus n_4 = 6$$

The Bouton theorem

Theorem (Bouton)

A $(n_1, n_2, \dots, n_k)$ position in the Nim game is a P-position if and only if $n_1 \oplus n_2 \oplus \dots \oplus n_N = 0$.

Proof

- Terminal states are P-positions This is obvious, the only terminal position is $(0, \dots, 0)$
- Positions such that $n_1 \oplus n_2 \oplus \dots \oplus n_N = 0$ go only to positions with Nim sum different from zero Suppose instead the new position is $(n_1', n_2, \dots, n_N)$ and $n_1' \oplus n_2 \oplus \dots \oplus n_N = 0 = n_1 \oplus n_2 \oplus \dots \oplus n_N$; then $n_1' = n_1$ (this should be proved!), which is impossible
- Positions with non null nim sum can go to a position with null Nim sum. Let $z := n_1 \oplus n_2 \oplus \dots \oplus n_N \neq 0$. Take a pile having 1 in the most left column where the expansion of z has 1, put there 0 and go right, leaving unchanged a digit corresponding to a 0 in the expansion of the sum, 1 otherwise. Easy to check that the so obtained number is smaller than the previous one. ■

An example

Example

From

$$\begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}$$

go to

$$\begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}$$

Observe: there are three initial good moves.

Conclusions

Games with perfect information can be "solved" by using backward induction

However backward induction is a concrete solution method only for very simple games, because of limited rationality.

According to conclusions we can reach different level of solutions:

1) Very weak solutions: not even the outcome is predictable (chess. . .)
2) Weak solutions: a logical argument provides the outcome, but how to reach it is not known (chomp, in general)
3) Solutions : categories of games where it is possible to produce the way to arrive to the rational outcome