# Fourth week

Roberto Lucchetti

LUISS

# Contents of the week

- Linear programming
- Zero sum games
- Conservative values
- Von Neumann theorem
- Fair games

# Linear programming problems



**Definition**

*A linear programming problem is the problem of maximizing or minimizing a linear function under linear inequality and equality constraints*

There are several forms of linear programming problems, and often there is a way to reduce a problem in some form to an equivalent problem in another form.

Linear programming has deep connections with game theory.

# Examples



$$\begin{cases} \min x_1 + x_2 \ : \\ x_1 + 2x_2 \geq 1 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

$$\begin{cases} \max y \ : \\ y \leq 1 \\ 2y \leq 1 \\ y \geq 0 \end{cases}$$

$$\begin{cases} \min x_1 + x_2 - x_3 \ : \\ x_1 + 2x_2 - x_3 \leq 11 \\ x_1 + x_2 + x_3 = 1 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

# Linear programming problem: first form



The linear programming problems are usually written in matrix form, and are "coupled":

## Definition

*The following two linear programming problems are said to be in duality:*

$$\begin{cases} \min x^t c : \\ x \geq 0, Ax \geq b \end{cases} \tag{1}$$

$$\begin{cases} \max y^t b : \\ y \geq 0, A^T y \leq c \end{cases} . \tag{2}$$

Here $A$ is an $m \times n$ matrix $b, c$ are vectors belonging to $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Observe that we denote by the same symbol 0 two vectors of different dimension: this does not create confusion since the entries of the vectors are all zero.

# Linear programming problem: second form



## Definition

*Let $A$ be an $m \times n$ matrix and let $b, c$ be vectors belonging to $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. The following two linear programming problems are said to be in duality:*

$$\begin{cases} \min x^t c : \\ Ax \geq b \end{cases} \tag{3}$$

$$\begin{cases} \max y^t b : \\ y \geq 0, A^T y = c \end{cases} . \tag{4}$$

## Exercise

*Show that the minimization problem in the second form can be written in an equivalent way in the first form; dualize this one and show that the dual is equivalent to the dual of the second form*

# Making explicit the problems



Let us write the primal problem making explicit vector-vector and vector-matrix products

$$\begin{cases} \min x^t c : \\ Ax \geq b \\ x \geq 0 \end{cases}$$

becomes

$$\begin{cases} \sum_{i=1}^n c_i x_i : \\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m \end{cases}$$

We can denote the $j$-th inequality as $(Ax - b)_j \geq 0$. Thus we have $n$ unknowns and, beyond the non negativity constraints, $m$ inequalities.

**Exercise**

*Write explicitly the constraint inequalities in all other linear problems*

# Relation between the values

Denote by

- $v$ the value of the primal min problem
- $V$ the value of the dual max problem.

**Theorem**

$$v \geq V.$$

**Proof** For the first type of problems:

$$x^t c \geq x^t A^t y = (x^t A^t y)^t = y^t Ax \geq y^t b$$

Since this is true for all admissible $x$ and $y$ the result holds in the first case.
In the second case

$$x^t c = x^t A^t y = y^t Ax \geq y^t b$$

# Feasibility



**Definition**

*The feasible set of a linear programming problem is the set of vectors fulfilling the linear inequalities/equalities of the problem*

Thus f.i. in the primal problem of the first type the feasible set is the set of vectors $x$ such that

$$\begin{cases} Ax \geq b \\ x \geq 0 \end{cases}$$

Easy examples show that, given two problems in duality,

- They can be both unfeasible
- Only one can be feasible
- Both can be feasible

# Example 1



Consider

$$\begin{cases} \min x_1 + x_2 \; : \\ x_1 + 2x_2 \geq 1 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Its dual is

$$\begin{cases} \max y \; : \\ y \leq 1 \\ 2y \leq 1 \\ y \geq 0 \end{cases}$$

Since $(x_1, x_2) = (0, \frac{1}{2})$ fulfills the constraints of the primal problem and $y = \frac{1}{2}$ fulfills the constraints of the dual problem, they are both feasible.

# Examples 2,3

Consider

$$\begin{cases} \min x_1 - x_2 : \\ x_1 + x_2 \geq 2 \\ -x_1 - x_2 \geq -1 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Its dual is

$$\begin{cases} \max 2y_1 - y_2 : \\ y_1 - y_2 \leq 1 \\ y_1 - y_2 \leq -1 \\ y \geq 0 \end{cases}$$

The primal problem is unfeasible (no $(x_1, x_2)$ fulfills the constraints), while $(n, n+1)$ fulfills the constraints of the dual problem for every $n$.

Taking $A = 0$, $b = (1, \dots, 1)$ and $c = (-1, \dots, -1)$ shows that both problems can be unfeasible.

# A first duality theorem



**Theorem**

*If one problem is unfeasible and the other is feasible, then the feasible is unbounded.*

Thus if the primal is unfeasible, and the dual is feasible, then the value $V$ of the dual problem is $V = +\infty$. Conversely, if the dual is unfeasible, and the primal is feasible, then the value $v$ of the primal problem is $v = -\infty$.

# The fundamental theorem of duality



The theorem in the previous slide shows that if one problem does not have solution then the other one is unfeasible, and conversely. The next one shows what happens when they are both feasible.

**Theorem**

*Suppose the two problems are both feasible. Then there are solutions $\bar{x}, \bar{y}$ of the two problems, and $\bar{x}^t c = \bar{y}^t b$.*

In other words, when they are both feasible they have both solution and also $V = v$. In this case we say that there is no duality gap.

**Corollary**

*If one problem is feasible and has solution, then also the dual problem is feasible and has solutions. Moreover there is no duality gap.*

# Complementarity conditions



Since

$$x^t c \geq x^t A^t y = y^t A x \geq y^t b$$

it follows that if $\bar{x}, \bar{y}$ are optimal,

$$\bar{x}^t c = \bar{x}^t A^t \bar{y} = \bar{y}^t A \bar{x} = \bar{y}^t b$$

This implies

$$\bar{x}^t (A^t y - c) = 0, \qquad \bar{y}^t (A\bar{x} - b) = 0$$

Since $\bar{x}, \bar{y} \geq 0$ and $Ax \geq b, A^t y \leq c$, it follows:

**Theorem**

*Let $\bar{x}, \bar{y}$ be solutions of the primal and dual problems. Then:*

$$\bar{x}_i > 0 \Longrightarrow \sum_{j=1}^{m} a_{ji} \bar{y}_j = c_i \qquad \bar{y}_j > 0 \Longrightarrow \sum_{i=1}^{n} a_{ij} \bar{x}_i = b_j$$

# An example



Consider

$$\begin{cases} \min x_1 + x_2 : \\ 2x_1 + x_2 \geq 2 \\ x_1 + 2x_2 \leq 2 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Its dual is

$$\begin{cases} \max 2y_1 - 2y_2 : \\ 2y_1 - y_2 \leq 1 \\ y_1 - 2y_2 \leq 1 \\ y_1 \geq 0, y_2 \geq 0 \end{cases}$$

We have $v = 1$, $(\bar{x}_1, \bar{x}_2) = (1, 0)$ $V = 1$, $(\bar{y}_1, \bar{y}_2) = (\frac{1}{2}, 0)$.

Check of the complementarity conditions:

$$\bar{y}_1 = \frac{1}{2} > 0 \Longrightarrow 2\bar{x}_1 + \bar{x}_2 = 2,\ \bar{x}_1 = 1 > 0 \Longrightarrow 2y_1 - y_2 = 1$$

# An example

$$\begin{cases} \min x_1 + 4x_2 : \\ x_1 \geq 1 \\ x_2 \leq 3 \\ x_1 + x_2 \leq 4 x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Its dual is

$$\begin{cases} \max y_1 - 2y_2 - 3y_3 : \\ y_1 - y_2 - y_3 \leq 1 \\ -y_3 \leq 4 \\ y_1 \geq 0, y_2 \geq 0 \end{cases}$$

The value of the problems is $v = V = 1$. Optimal solutions: for the primal $(1, x_2) : 0 \leq x_2 \leq 2$, for the dual $y = (1, 0, 0)$

**Remark**

*The feasible and the solution set are always convex, a special type of convex set: the smallest convex containing a finite number of points, called the extreme points of the convex, and a solution can be always found by checking the extreme points of the feasible set.*

**Exercise**

*Check the complementarity conditions.*

# General form



An interesting case of non cooperative game is is when there are two players, with opposite interests.

**Definition**

*A two player zero sum game in strategic form is the triplet* $(X, Y, f : X \times Y \to \mathbb{R})$

Conventionally $f(x, y)$ is what Player I gets from Player II, when they play $x$, $y$ respectively.

# Finite game



In the finite case $X = \{1, 2, \dots, n\}$, $Y = \{1, 2, \dots, m\}$ the game is described by a payoff matrix $P$

**Example**

$$P = \begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}$$

Player I selects row $i$, Player II selects column $j$.

# A different approach to solve them



$$\begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}.$$

Player I can guarantee herself to get at least

$$v_1 = \max_i \min_j p_{ij}$$

Player II can guarantee himself to pay no more than

$$v_2 = \min_j \max_i p_{ij}$$

$\min_j p_{1j} = 1$, $\min_j p_{2j} = 5$, $\min_j p_{3j} = 0$ $\quad v_1 = 5$
$\min_i p_{i1} = 8$, $\min_j p_{i2} = 5$, $\min_j p_{i3} = 8$, $\quad v_2 = 5$

Rational outcome 5. Rational behavior ($\bar{\imath} = 2, \bar{\jmath} = 2$).

# Alternative idea of solution

Suppose

- $v_1 = v_2 := v$
- $\bar{\imath}$ is the row such that $p_{\bar{\imath}j} = \max_i \min_j p_{ij} = v$ so that for all $j$ $p_{\bar{\imath}j} \geq v$
- $\bar{\jmath}$ is the column such that $p_{i\bar{\jmath}} = \min_j \max_i p_{ij} = v$ so that for all $i$ $p_{i\bar{\jmath}} \leq v$

Then $p_{\bar{\imath}\bar{\jmath}} = v$ and $p_{\bar{\imath}\bar{\jmath}} = v$ is the rational outcome of the game

**Remark**

- $\bar{\imath}$ *is an optimal strategy for Player I, because he cannot get more than $v$, since $v$ is the conservative value of Player II*
- $\bar{\jmath}$ *is an optimal strategy for Player II, because he cannot pay less than $v$, since $v$ is the conservative value of Player I*

# For arbitrary games

$(X, Y, f : X \times Y \to \mathbb{R})$

The players can guarantee to themselves (almost):

Player I: $v_1 = \sup_x \inf_y f(x, y)$

PLAYER II: $v_2 = \inf_y \sup_x f(x, y)$

$v_1, v_2$ are the conservative values of the players

# Optimality



Suppose $v_1 = v_2 := v$ , strategies $\bar{x}$ and $\bar{y}$ exist such that

$$f(\bar{x}, y) \geq v, \quad f(x, \bar{y}) \leq v$$

for all $y$ and for all $x$

Then

- $\bar{x}$ is an optimal strategy for Player I
- $\bar{y}$ is an optimal strategy for Player II
- $f(\bar{x}, \bar{y}) = v$ is the rational outcome of the game

# $v_1 \leq v_2$



## Proposition

*Let $X, Y$ be any sets and let $f : X \times Y \to \mathbb{R}$ be an arbitrary function. Then*

$$\sup_x \inf_y f(x,y) \leq \inf_y \sup_x f(x,y)$$

**Proof** Observe that, for all $x, y$,

$$\inf_y f(x,y) \leq f(x,y) \leq \sup_x f(x,y)$$

Thus

$$\inf_y f(x,y) \leq \sup_x f(x,y)$$

Since the left hand side of the above inequality does not depend on $y$ and the right hand side on $x$, the thesis follows ■

In every game $v_1 \leq v_2$, as expected

# Equality need not hold



**Example**

$$P = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

$v_1 = -1, \; v_2 = 1$

Nothing unexpected...

# Case $v_1 < v_2$

Finite case: mixed strategies. Game: $n \times m$ matrix $P$.

Strategy space for Player I:

$$\Sigma_n = \{x = (x_1, \dots, x_n) : x_i \geq 0, \sum_{i=1}^{n} x_i = 1\}$$

Strategy space for Player II:

$$\Sigma_m = \{y = (y_1, \dots, y_m) : y_j \geq 0, \sum_{j=1}^{m} y_j = 1\}$$

$$f(x, y) = \sum_{i=1,\dots,n, j=1,\dots,m} x_i y_j p_{ij} = x^t P y$$

The mixed extension of the initial game $P$: $(\Sigma_n, \Sigma_m, f(x, y) = x^t P y)$

# To prove existence of a rational outcome



What must be proved, to have existence of a rational outcome:

1. $v_1 = v_2$
2. there exists $\bar{x}$ fulfilling
$$v_1 = \sup_x \inf_y f(x,y) = \inf_y f(\bar{x},y)$$
3. there exists $\bar{y}$ fulfilling
$$v_2 = \inf_y \sup_x f(x,y) = \sup_x f(x,\bar{y})$$

In the finite case $\bar{x}$ and $\bar{y}$ fulfilling 1) and 2) always exist; thus existence is equivalent to 1)

# The von Neumann theorem



**Theorem**

*A two player, finite, zero sum game as described by a payoff matrix $P$ has a rational outcome: the two conservative values of the players agree and there are optimal strategies $\bar{x}$, $\bar{y}$ for the players.*

**Remark**

*We remind that when the two conservative values agree the strategy $\bar{x}$ is optimal for Player I if and only if it guarantees her to get the (common conservative) value no matter what Player II does; dually the strategy $\bar{y}$ is optimal for Player II if and only if it guarantees him to get the (common conservative) value no matter what Player I does.*

# Finding optimal strategies:Player I



Player I must choose a probability distribution $\Sigma_n \ni x = (x_1, \dots, x_n)$:

$$
\begin{aligned}
& x_1 p_{11} + \cdots + x_n p_{n1} \geq v \\
& \ldots \\
& x_1 p_{1j} + \cdots + x_n p_{nj} \geq v \\
& \ldots \\
& x_1 p_{1m} + \cdots + x_n p_{nm} \geq v
\end{aligned}
$$

where $v$ must be as large as possible

# Finding optimal strategies: Player II



Player II must choose a probability distribution $\Sigma_m \ni y = (y_1, \dots, y_m)$:

$$
\begin{aligned}
&y_1 p_{11} + \dots + y_m p_{1m} \leq w \\
&\dots \\
&y_1 p_{i1} + \dots + y_m p_{im} \leq w \\
&\dots \\
&y_1 p_{n1} + \dots + y_m p_{nm} \leq w
\end{aligned}
$$

where $w$ must be as small as possible

# In matrix form

Player I:

$$\begin{cases} \max_{x,v} v : \\ P^t x \geq v 1_m \\ x \geq 0 \quad 1^t x = 1 \end{cases} \tag{5}$$

Player II:

$$\begin{cases} \min_{y,w} w : \\ P y \leq w 1_n \\ y \geq 0 \quad 1^t y = 1 \end{cases} \tag{6}$$

# A more familiar form

We can suppose that all $p_{ij} > 0$ (without loss of generality) Thus $v > 0$.

Set $\alpha_i = \frac{\alpha_i}{v}$.Condition $\sum_{i=1}^{n} x_i = 1$ becomes $\sum_{i=1}^{n} \alpha_i = \frac{1}{v}$. Thus maximizing $v$ is equivalent tominimizing $\sum_{i=1}^{n} \alpha_i$.

Thus, in matrix form:

$$\begin{cases} \min \alpha^t 1_n : \\ \alpha \geq 0, P^t \alpha \geq 1_m \end{cases} . \tag{7}$$

The dual

$$\begin{cases} \max \beta^t 1_m : \\ \beta \geq 0, P\beta \leq 1_n \end{cases} . \tag{8}$$

Exactly the optimal problem for Player II!

# The complementarity conditions

The complementarity conditions become

- $\beta_j > 0 \Longrightarrow \sum_{i=1}^{n} p_{ji}\alpha_i = 1$, i.e. $y_j > 0 \Longrightarrow \sum_{i=1}^{n} p_{ji}x_i = v$
- $\alpha_i > 0 \Longrightarrow \sum_{j=1}^{m} p_{ij}\beta_j = 1$, i.e. $x_i > 0 \Longrightarrow \sum_{j=1}^{m} p_{ji}y_j = v$

Since $\sum_{i=1}^{n} p_{ji}x_i$ is the expected value for Player II if she plays column $j$ and Player I the mixed strategy $x = (x_1, \dots, x_n)$, the complementarity conditions show, one more time, that if one Player plays with positive probability a pure strategy, this must be optimal.

# Summarizing



A finite zero sum game has always rational outcome in mixed strategies

The set of optimal strategies for the players is a nonempty closed convex set, the smallest convex set containing a finite number of points, called the extreme points of the set

The outcome, at each pair of optimal strategies, is the common conservative value $v$ of the players

# The Nash equilibria of a zero sum game



**Theorem**

*Let $X$, $Y$ be (nonempty) sets and $f : X \times Y \to \mathbb{R}$ a function. Then the following are equivalent:*

1. *The pair $(\bar{x}, \bar{y})$ is a Nash equilibrium, i.e. fulfills*

$$f(x, \bar{y}) \leq f(\bar{x}, \bar{y}) \leq f(\bar{x}, y) \quad \forall x \in X, \ \forall y \in Y$$

2. *The following conditions are satisfied:*
   *(i)* $\inf_y \sup_x f(x, y) = \sup_x \inf_y f(x, y)$*: the two conservative values do agree*
   *(ii)* $\inf_y f(\bar{x}, y) = \sup_x \inf_y f(x, y)$*: $\bar{x}$ is optimal for Player I*
   *(iii)* $\sup_x f(x, \bar{y}) = \inf_y \sup_x f(x, y)$*: $\bar{y}$ is optimal for Player II*

# Proof

**Proof** 1) implies 2). From 1) we get:

$$\inf_y \sup_x f(x,y) \leq \sup_x f(x,\bar{y}) = f(\bar{x},\bar{y}) = \inf_y f(\bar{x},y) \leq \sup_x \inf_y f(x,y)$$

Since $v_1 \leq v_2$ always holds, all above inequalities are equalities

Conversely, suppose 2) holds Then

$$\inf_y \sup_x f(x,y) \overset{(iii)}{=} \sup_x f(x,\bar{y}) \geq f(\bar{x},\bar{y}) \geq \inf_y f(\bar{x},y) \overset{(ii)}{=} \sup_x \inf_y f(x,y)$$

Because of (i), all inequalities are equalities and the proof is complete
$\blacksquare$

# As a consequence of the theorem



- Any $(\bar{x}, \bar{y})$ Nash equilibrium of the zero sum game provides optimal strategies for the players
- Any pair of optimal strategies for the players provides a Nash equilibrium for the zero sum game

Thus Nash theorem is a generalization of von Neumann theorem

# A comment

## Remark

*Von Neumann approach with conservatives values shows that, in the particular case of the zero sum game:*

- *Players can find their optimal behavior independently for the other players*
- *Any pair of optimal strategies provides a Nash equilibrium; this implies no need of coordination to reach an equilibrium*
- *Every Nash equilibrium provides the same utility (payoff) to the players: multiplicity of solutions does not create problems*
- *Nash equilibria are easy to be found in zero sum games.*

# Symmetric games



**Definition**

*A square matrix $n \times n$ $P = (p_{ij})$ is said to be antisymmetric provided $p_{ij} = -p_{ji}$ for all $i, j = 1, \ldots, n$. A (finite) zero sum game is said to be fair if the associated matrix is antisymmetric*

In fair games there is no favorite player

# Fair outcome



## Proposition

*In a fair game*

- *the value is* 0
- $\bar{x}$ *is an optimal strategy for Player I if and only if it is optimal for Player II*

**Proof** Since

$$x^t P x = (x^t P x)^t = x^t P^t x = -x^t P x,$$

$f(x, x) = 0$ for all $x$, thus $v_1 \leq 0, v_2 \geq 0$

Then $v = 0$.

If $\bar{x}$ is optimal for the Player I, $\bar{x}^t P y \geq 0$ for all $y$

Thus $y^t P \bar{x} \leq 0$ for all $y \in \Sigma_n$, and $\bar{x}$ is optimal for Player II ■

# Finding optimal strategies in a fair game

Need to solve the system of inequalities

$$\begin{array}{l} x_1 p_{11} + \cdots + x_n p_{n1} \geq 0 \\ \dots \\ x_1 p_{1j} + \cdots + x_n p_{nj} \geq 0 \\ \dots \\ x_1 p_{1m} + \cdots + x_n p_{nm} \geq 0 \end{array}$$

with the extra conditions:

$$x_i \geq 0, \qquad \sum_{i=1}^{n} x_i = 1$$

# A proposed exercise



### Exercise

*Find the optimal strategies of the players in the rock,scissors, paper game and in the following fair game:*

$$P = \begin{pmatrix} 0 & 3 & -2 & 0 \\ -3 & 0 & 0 & 4 \\ 2 & 0 & 0 & -3 \\ 0 & -4 & 3 & 0 \end{pmatrix}$$