Second week

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Contents of the week

- Pure strategies vs mixed strategies best response multifunction
- Nash Theorem (idea of the proof)
- Calculus of equilibrium in mixed strategies in simple finite games
- Examples of n-person finite games: Braess paradox, Hotelling game, El Farol bar, Cournot model of duopoly, auctions. . .

A Nash equilibrium need not to exist

Consider the game:

$$\begin{pmatrix} (4,0) & (3,1) \\ (3,5) & (5,0) \end{pmatrix}$$

There is no Nash equilibrium!

Thus a player cannot play the same strategy at any time; this could be observed and other players could take advantage from this.

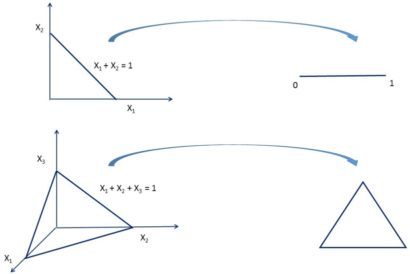
- It makes sense to play strategies according to some probability scheme;
- The probabilities must be chosen strategically!

Simplexes

Definition

Let A be a finite strategy set with l elements(called the set of pure strategies. The set of the mixed strategies over the set A is the fundamental simplex

$$\Sigma_l = \{x = (x_1, \dots, x_l) : x_i \geq 0, \sum_{i=1}^{l} x_1 = 1\}$$



Observe that Σ_l is a $l - 1$-dimensional space

Finite games, mixed strategies

Take a finite game (A, B) (A, B $n \times m$ matrices), where Pl 1 has $I = \{1, \dots, i, \dots, n\}$ as strategy set, Pl 2 $J = \{1, \dots, i, \dots, m\}$. The mixed extension $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$ of the game is defined as:

- $X = \Sigma_n$ is the strategy space of Pl 1
- $Y = \Sigma_m$ is the strategy space of Pl 2
- $f(x, y) = \sum_{i=1,\dots,n, j=1,\dots,m} x_i y_j a_{ij} = x^t Ay$ is the utility function of Pl 1
- $g(x, y) = \sum_{i=1,\dots,n, j=1,\dots,m} x_i y_j b_{ij} = x^t By$ is the utility function of Pl 2

The rows/columns are the strategies of the players in the reference game. In the mixed extension, when the players choose $x \in \Sigma_n$ and $y \in \Sigma_m$ the outcome ij in the initial game has probability $x_i y_j$ to occur. Thus $f(x, y)$, $g(x, y)$ above are the expected utilities when the players play x and y respectively.

Equivalent formulation of utilities

- The expected payoff for player 1 when playing the pure strategy $i \in I$ against the mixed strategy $y \in \Sigma_J$ of player 2 when playing the pure strategy $i \in I$ against the mixed strategy $y \in \Sigma_J$ of player 2 is

$$u_i(y) = \sum_{j=1}^{m} a_{ij} y_j$$

and then

$$f(x,y) = \sum_{i=1}^{n} x_i u_i(y)$$

- The expected payoff for player 2 when playing the pure strategy $j \in J$ against against the mixed strategy $x \in \Sigma_I$ of player 1 is

$$v_j(x) = \sum_{i=1}^{n} b_{ij} x_i$$

and then

$$g(x,y) = \sum_{j=1}^{m} y_j v_j(x)$$

Best reaction

The idea of Nash equilibrium relies on the fact that players maximize their utility functions with respect to their own variable and knowing, or taking for granted the choice of the other player(s)

This leads to the idea of the best reaction (multifunction)

$$BR_1 : Y \to X : \quad BR_1(y) = \{x_0 \in X : f(x_0, y) \geq f(x, y) \, \forall x \in X\}$$

$$BR_2 : X \to Y : \quad BR_2(x) = \{y_0 \in Y : g(x, y_0) \geq f(x, y) \, \forall y \in Y\}$$

In words, to every $y \in Y$ ($x \in X$) BR_1 (BR_2) associates the set of the x_0 maximizing the function $f(\cdot, y)$ (y_0 maximizing the function $g(x, \cdot)$).

It is then clear that
$(\bar{x}, \bar{y})$ is a N.E. profile if and only if $(\bar{x}, \bar{y}) \in (BR_1(\bar{y}), BR_2(\bar{x}))$

Best responses: equivalent formulation

Player 1: $\max_{x\in\Sigma_n} \sum_{i=1}^{n} x_i u_i(y) \Rightarrow BR_1(y) = \operatorname*{Argmax}_{x\in\Sigma_I} \sum_{i=1}^{n} x_i u_i(y)$

Player 2: $\max_{y\in\Sigma_m} \sum_{j=1}^{m} y_j v_j(x) \Rightarrow BR_2(x) = \operatorname*{Argmax}_{y\in\Sigma_J} \sum_{j=1}^{m} y_j v_j(x)$

The Nash theorem

The following is the Nash theorem about existence of equilibria

Theorem

Given the game $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$, *suppose:*

- X *and* Y *are closed bounded convex subsets of some finite dimensional vector space*
- f, g *continuous*
- $x \mapsto f(x, y)$ *is a concave function for all fixed* $y \in Y$
- $y \mapsto g(x, y)$ *is a concave function for all fixed* $x \in X$

Then the game has an equilibrium

The proof relies on showing that the best reaction multifunction has a fixed point, i.e. a point $(\bar{x}, \bar{y})$ such that $(\bar{x}, \bar{y}) \in (BR_1(\bar{y}), BR_2(\bar{x}))$

The finite case

Corollary

A finite game admits always a Nash equilibrium in mixed strategies

- X and Y are simplexes, hence closed bounded and convex subsets of some finite vector space
- $f(x,y) = x^t Ay, g(x,y) = x^t By$ are continuous functions, and concave with respect to every variable, when the other is fixed

Thus the assumption of the theorem are fulfilled.

Linearity of the problem

Since

$$\text{Player 1:} \quad \max_{x \in \Sigma_n} \sum_{i=1}^{n} x_i u_i(y) \Rightarrow BR_1(y) = \operatorname*{Argmax}_{x \in \Sigma_I} \sum_{i=1}^{n} x_i u_i(y)$$

$$\text{Player 2:} \quad \max_{y \in \Sigma_m} \sum_{j=1}^{m} y_j v_j(x) \Rightarrow BR_2(x) = \operatorname*{Argmax}_{y \in \Sigma_J} \sum_{j=1}^{m} y_j v_j(x)$$

we have

Remark

Once fixed the strategies of the other players, the utility function of one player is linear in its own variable

Finding Nash equilibria

The game:

$$\begin{pmatrix} (1,0) & (0,3) \\ (0,2) & (1,0) \end{pmatrix}$$

Pl1 playing $(p, 1-p)$, PL2 playing $(q, 1-q)$:

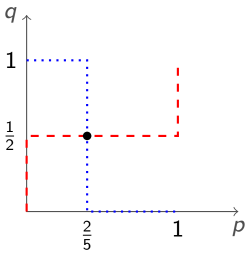
$$f(p,q) = pq + (1-p)(1-q) = p(2q-1) - q + 1$$

$$g(p,q) = 3p(1-q) + 2(1-p)q = q(2-5p) + 3p$$

The best reply multifunctions

$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad 0 \le q \le \frac{1}{2} \\ p \in [0,1] & \text{if} \quad q = \frac{1}{2} \\ p = 1 & \text{if} \quad q > \frac{1}{2} \end{cases}$$

$$BR_2(p) = \begin{cases} q = 1 & \text{if} \quad 0 \le p \le \frac{2}{5} \\ q \in [0,1] & \text{if} \quad p = \frac{2}{5} \\ q = 0 & \text{if} \quad p > \frac{2}{5} \end{cases}$$



Equivalent calculation

From the point of view of Pl1, if Pl 2 plays $(q, 1-q)$

- expected value form the first row: q
- expected value form the second row: $1-q$

Thus

- First row is the unique optimal response if $q > \frac{1}{2}$
- Second row is the unique optimal response if $q < \frac{1}{2}$
- Both rows optimal response if $q = \frac{1}{2}$, and thus any mixture between the two is still optimal

Maximizing over a simplex

In the above example we saw that among the best reactions of a player to a given strategy of the other one there is always at least one pure strategy. This happens always. Let us see why.

Consider the problem of maximizing a weighted average

$$\begin{aligned}&\text{maximize } \ 15x_1 + 18x_2 + 23x_3 + 23x_4\\&\text{subject to:}\\&x_1 + x_2 + x_3 + x_4 = 1\\&x_1, \dots, x_4 \geq 0\end{aligned}$$

Clearly the optimal value is 23 and is attained by putting all the weight on the variables x_3 and x_4 which have larger coefficients.

In particular $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ are optimal, as well as $(0, 0, \frac{1}{2}, \frac{1}{2})$.

In general the optimal solutions are all vectors (x_1, x_2, x_3, x_4) with $x_1 = x_2 = 0$ and $x_3, x_4 \geq 0$ with $x_3 + x_4 = 1$.

Maximizing over a simplex

Consider Pl1 and fix a mixed strategy y of PL2. Let $u_i = u_i(y)$ be the expected payoff of Pl1 when playing $i \in I$. Then the best response requires to solve

$$\max_{x \in \Sigma_n} \sum_{i=1}^{n} x_i u_i$$

In order to render the weighted average $\sum_{i=1}^{n} x_i u_i$ as large as possible one simply has to put all the weight on the variables with largest coefficients u_i.

Thus, setting $v = \max_{i \in I} u_i$ we actually have

$$\max_{x \in \Sigma_n} \sum_{i=1}^{n} x_i u_i = v$$

and $x \in \Sigma_n$ is a best response if and only if

$$(\forall i \in I) \; x_a > 0 \Rightarrow u_a = v.$$

Note that there is always a best response in pure strategies: choose any a with maximal u_a and set $x_a = 1$ and $x_{a'} = 0$ for $a' \neq a$.

Maximizing over a simplex

Definition

For a non-negative vector $(x_1, \ldots, x_n) \geq 0$ we define its support as the set of indexes of its strictly positive entries, that is

$$spt(x) = \{i : x_i > 0\}$$

Indifference

Remark

Suppose $(\bar{x}, \bar{y})$ is a NE in mixed strategies. Suppose spt $\bar{x} = \{1, \dots, k\}$, spt $\bar{y} = \{1, \dots, l\}$, and $f(\bar{x}, \bar{y}) = v$. Then it holds:

$$\begin{cases} a_{11}\bar{y}_1 + a_{12}\bar{y}_2 + \dots + a_{1l}\bar{y}_l & = v \\ \dots & = v \\ a_{k1}\bar{y}_1 + a_{k2}\bar{y}_2 + \dots + a_{kl}\bar{y}_l & = v \\ a_{(k+1)1}\bar{y}_1 + a_{(k+1)2}\bar{y}_2 + \dots + a_{(k+1)l}\bar{y}_l & \leq v \\ \dots & \leq v \\ a_{n1}\bar{y}_1 + a_{n2}\bar{y}_2 + \dots + a_{nl}\bar{y}_l & \leq v \end{cases}$$

Brute force algorithm

1. Guess the supports of the equilibria $spt(\bar{x})$ and $spt(\bar{y})$
2. Ignore the inequalities and find x, y, v, w by solving the linear system of $n + m + 2$ equations

$$\begin{cases} \sum_{i=1}^{n} x_i = 1 & \\ \sum_{j=1}^{m} a_{ij} y_j = v & \text{for all } i \in spt(\bar{x}) \\ x_i = 0 & \text{for all } i \notin spt(\bar{x}) \end{cases}$$

$$\begin{cases} \sum_{j=1}^{m} y_j = 1 & \\ \sum_{i=1}^{n} b_{ij} x_i = w & \text{for all } j \in spt(\bar{y}) \\ y_j = 0 & \text{for all } j \notin spt(\bar{y}) \end{cases}$$

3. Check whether the ignored inequalities are satisfied. If $x_i \geq 0, y_j \geq 0, \sum_{j=1}^{m} a_{ij} y_j \leq v$ and $\sum_{i=1}^{n} b_{ij} x_i \leq w$ then Stop: we have found a mixed equilibrium. Otherwise, go back to step 1 and try another guess of the supports.

A simple case

The above indifference principle simplifies if one looks for existence of fully mixed Nash equilibria, i.e. those equilibria where all rows/columns are played with positive probability.

Suppose $(\bar{x}, \bar{y})$ is such an equilibrium. Then it holds that

$$a_{i1}\bar{y}_1 + a_{i2}\bar{y}_2 + \cdots + a_{im}\bar{y}_m = a_{k1}\bar{y}_1 + a_{k2}\bar{y}_2 + \cdots + a_{km}\bar{y}_m$$

for all $i, k = 1, \dots n$, and similarly

$$b_{1r}\bar{x}_1 + b_{2r}\bar{x}_2 + \cdots + b_{nr}\bar{x}_n = b_{1s}\bar{x}_1 + b_{2s}\bar{x}_2 + \cdots + b_{ns}\bar{x}_n$$

for all $r, s = 1, \dots m$ with the further conditions

$$p_j, q_j \geq 0, \quad \sum_{i=1}^{n} p_i = 1, \quad \sum_{j=1}^{m} q_j = 1$$

Revisiting an example

The game:

$$\begin{pmatrix} (1,0) & (0,3) \\ (0,2) & (1,0) \end{pmatrix}$$

Since no equilibria in pure strategies, apply the indifference principle: calling $(q, 1-q)$ the equilibrium strategy of Pl2 it holds $q = 1 - q$, providing $q = \frac{1}{2}$. Analogously calling $(p, 1-p)$ the equilibrium strategy of Pl1 it holds $2 - 2p = 3p$, providing $p = \frac{2}{5}$.

Another example

Let $X = Y = [0, 10]$ and let the utilities of the players be $f(x, y) = -x^2 - 2xy + 8x + 1$, $g(x, y) = -y^2 - 2xy + 8y + 7$.

The functions are continuous and concave in one variable when the other one is fixed. The common strategy set is a closed interval, thus a closed convex bounded set.

Since

$$\begin{cases} f_x(x, y) = -2x - 2y + 8 \\ g_y(x, y) = -2y - 2x + 8 \end{cases}$$

$$BR_1(y) = \begin{cases} 4 - y & \text{if} \quad y \leq 4 \\ 0 & \text{if} \quad y > 4 \end{cases}$$

$$BR_2(x) = \begin{cases} 4 - x & \text{if} \quad x \leq 4 \\ 0 & \text{if} \quad x > 4 \end{cases}$$

Thus there is only Nash equilibrium $(\bar{x}, \bar{y}) = (2, 2)$.

General strategic games

Consider an n-player game with strategy sets X_i and payoffs $f_i : X \to \mathbb{R}$ where as usual $X = \prod_{j=1}^{n} X_j$ is the set of strategy profiles.

Each player $i = 1, \dots, n$ maximizes her payoff with respect to her own variable $x_i \in X_i$ while taking for granted the choice of the other players $x_{-i} \in \prod_{j \neq i} X_j$.

Define the best response map $BR_i : X_{-i} \to X_i$ as

$$BR_i(x_{-i}) = \{x_i \in X_i : f_i(x_i, x_{-i}) \geq f(z_i, x_{-i}) \, \forall z_i \in X_i\}$$

which associates to each possible strategies x_{-i} of the other players the set of x_i's that maximize the payoff $f(\cdot, x_{-i})$.

Then: $(\bar{x}_i)_{i=1}^{n}$ is a Nash equilibrium if and only if for each player $i = 1, \dots, n$ we have $\bar{x}_i \in BR_i(\bar{x}_{-i})$.

The Nash theorem

Theorem

Given a n-player game with strategy sets X_i and payoff functions $f_i : X \to \mathbb{R}$ where $X = \prod_{i=1}^{n} X_i$. Suppose:

- *each X_i is a closed bounded convex subset in a finite dimensional space $\mathbb{R}^{d_i}$*
- *each $f_i : X \to \mathbb{R}$ is continuous*
- *$x_i \mapsto f_i(x_i, x_{-i})$ is a (quasi) concave function for each fixed $x_{-i} \in X_{-i}$*

Then there exists at least one Nash equilibrium.

Mixed equilibria for n-player finite games

Consider an n-person finite game with strategy sets A_i and payoffs $f_i(a_1, \dots, a_n)$.
In the *mixed extension* each player i choses a probability distribution $x^i \in \Sigma_{A_i}$, that is to say, $x^i_{a_i} \geq 0$ for all $a_i \in A_i$ and $\sum_{a_i \in A_i} x^i_{a_i} = 1$.

Denote $A = \prod_{i=1}^n A_i$ the set of pure strategy profiles. The probability of observing an outcome $(a_1, \dots, a_n) \in A$ is the product $\prod_{i=1}^n x^i_{a_i}$ and the *expected* payoffs are:

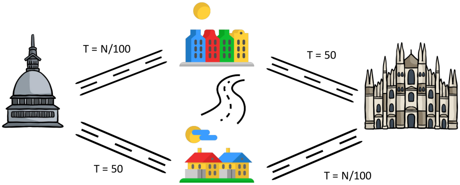
$$\bar{f}_i(x^1, \dots, x^n) = \sum_{(a_1, \dots, a_n) \in A} f_i(a_1, \dots, a_n) \prod_{j=1}^n x^j_{a_j} = \sum_{a_i \in A_i} x^i_{a_i} u_i(a_i, x^{-i})$$

$$u_i(a_i, x^{-i}) = \sum_{a_j \in A_j, j \neq i} f_i(a_1, \dots, a_n) \prod_{j \neq i} x^j_{a_j}$$

Corollary

Every n-player finite game has at least one Nash equilibrium in mixed strategies.

First example: the Braess paradox



4.000 people travel form one city to another one. Every player wants to minimize time. N is the number of people driving in the corresponding road

What are the Nash equilibria? What happens if the North-South street between the two small cities is made available to cars and time to travel on it is 5 minutes?

El Farol bar

In Santa Fe there are 500 young people, happy to go to the El Farol bar. More people in the bar, happier they are, till they reach 300 people. They can also choose to stay at home. So utility function can be assumed to be 0 if they stay at home, $u(x) = x$ if $x \leq 300$, $u(x) = 300 - x$ if $x > 300$.

Essentially there is a Nash equilibrium, where 300 young people are in the bar, the other stay at home. An asymmetric situation, notwithstanding the players are symmetric.

A mixed symmetric equilibrium is present in this case.

Hotelling game

In a beach one km long and with people uniformly distributed, some ice cream vendors must place their cart. People go to the closest cart to buy ice cream. People having some carts at the same distance split among the carts

The following holds

1. If there are only two ice cream vendors, the unique equilibrium is when they both stay in the center of the beach;
2. When there are three, no Nash equilibrium exists;
3. When they are four or five, there is one equilibrium, up to permutations of the carts
4. There are infinite equilibria when the carts are six or more

Auctions

Auctions, several types, since ancient times...

1. Sequential offers
2. Sealed
3. First price
4. Second price
5. Different termination rules
6. ...

Auctions: description

- There are n bidders, each one has a valuation v of the object, and suppose $v_1 > v_2 > \cdots > v_n$
- Each bidder proposes a (non negative) bid, seen as strategy for the player. Thus the strategy space of the players is $[0, \infty)$
- An assignment rule for the payment must be decided, including the rule to handle the ties
- Player i gets utility $v_i - b_i$ if he wins, zero otherwise.

We consider only auctions where the winner is the highest bid and we assume that in case of tie of the highest bid the winner is the player evaluating more the object.

First price auction

For the first price auction the payment rule is: the winner j offering the bid b_j pays her bid. Other players pay nothing

1. for Player i bidding more than v_i is weakly dominated
2. One Nash equilibrium is $(v_2, v_2, v_3, \ldots, v_n)$
3. In all equilibria the winner is Player One
4. The two highest bids are the same and one is made by Player One. The highest bid b_1 satisfies $v_2 \leq b_1 \leq v_1$. All such bid profiles are Nash equilibria

Second price auctions

For the second price auction the payment rule is: the winner j offering the bid b_j pays the second best bid. Other players pay nothing,

1. One Nash equilibrium is $(v_1, v_2, v_3, \dots, v_n)$
2. Other equilibria: $(v_1, 0, 0, \dots, 0)$, $(v_2, v_1, v_3, \dots, v_n)$
3. A player's bid equalizing her evaluation is a weakly dominant strategy

Duopoly models

Two firms choose quantities of a good to produce. Firm 1 produces quantity q_1, firm 2 produces quantity q_2, the unitary cost of the good is $c > 0$ for both firms. A quantity $a > c$ of the good saturates the market. The price $p(q_1, q_2)$ is

$$p = \max\{a - (q_1 + q_2), 0\}$$

Payoffs:

$$\max\{u_1(q_1, q_2) = q_1 p(q_1, q_2) - cq_1 = q_1(a - (q_1 + q_2)) - cq_1, 0\}$$

$$\max\{u_2(q_1, q_2) = q_2 p(q_1, q_2) - cq_2 = q_2(a - (q_1 + q_2)) - cq_2, 0\}$$

The monopolist

Suppose $q_2 = 0$. Thus $u_1(q_1, 0) = \max\{-q_1^2 + (a-c)q_1, 0\}$.

Since $-q_1^2 + (a-c)q_1 < 0$ if $q_1 > a-c$, we can suppose $q_1 \leq a-c$, thus the utility function is concave and then, making the first derivative to vanish we get

$$q_M = \frac{a-c}{2}, \quad p_M = \frac{a+c}{2} \quad u_M(q_M) = \frac{(a-c)^2}{4}$$

The duopoly

Since the partial derivative of $u_1(q_1, q_2) \leq 0$ for every $q_2 > 0$, if and only if $q_1 > \frac{a-c}{2}$, and by symmetry, we can suppose $q_1, q_2 \leq \frac{a-c}{2}$, thus the utility functions are concave in $[0, \frac{a-c}{2}]$ and then, making the first derivative to vanish we get

$$a - 2q_1 - q_2 - c = 0, \qquad a - 2q_2 - q_1 - c = 0,$$

$$q_i = \frac{a-c}{3}, \quad p = \frac{a+2c}{3} \quad u_i(q_i) = \frac{(a-c)^2}{9}.$$

Exercise with bonus: prove that the N.E. can be found by eliminating strictly dominated strategies (this requires infinitely many steps. . .)