First week

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- Players, strategies, payoffs
- Basic assumptions of the theory
- Elimination of dominated strategies
- Nash Equilibrium

Setting

Optimization

1. One decision maker
2. At least two decision makers

Possible variants with one decision maker: scalar/vector optimization, deterministic/stocastic ...

Many possible variants with many decision makers

Game theory, Social choice, Mechanism design

Crucial difference:

The best to do is easily definable with one decision maker, much more difficult with many decision makers

Loose Description of game

A process that can be described by:

1. A set of players (with more than one element)
2. An initial situation
3. The way the players must act and all their available moves
4. All possible final situations
5. The preferences of all agents on the set of the final situations

Examples:

1. The chess game
2. Two people bargaining how to divide a pie
3. People sharing a common resource
4. Establishing fees for a common resource
5. ...

Games are efficient models for an enormous amount of everyday life situations

Modeling the game

A game can be modeled by specifying

- The set of the players
- Their strategies
- Their preferences on all possible outcomes of the game

A strategy for a player: the specification of an action at any time she could be called to make a move

Assumptions of the theory

Players are

1. Egoistic
2. Rational

Egoistic means that the player tries to get the best for her

Observe this is NOT an ethical issue, but a mathematical assumption

Rationality is a much more involved issue

Preferences

Definition

Let X be a set. A preference relation on X is a binary relation $\succeq$ fulfilling, for all $x, y, z \in X$:

1) $x \succeq x$ (*reflexivity*)

2) *either* $x \succeq y$ *or* $y \succeq x$ (*completeness*)

3) $x \succeq y \wedge y \succeq z$ *imply* $x \succeq z$ (*transitivity*)

The first rationality assumption reads:

The agents are able to provide a preference relation over the outcomes of the game.

Utility functions

Definition

Let $\succeq$ be a preference relation over X. A ***utility function*** *representing $\succeq$ is a function $u : X \to \mathbb{R}$ such that*

$$u(x) \geq u(y) \iff x \succeq y.$$

1. A utility function need not to exist, however it exists in general setting, in particular if X is a finite set
2. When a utility function exists, then infinite utility functions do exist

Utility functions are useful in order analyze the game, to have a numerical representation of the preferences.

They are needed any time some random choice is present in the game.

Equivalent utilities

If $u(\cdot)$ is a utility function of one player, then $g \circ u$ is another utility function, for every strictly increasing function g.

Thus $u^3(\cdot), e^{u(\cdot)}, \arctan(u(\cdot))$ are other utility functions.

Given a utility function $u(\cdot)$, a widely used transformation is any function $v(\cdot) = au(\cdot) + b$, where $a > 0$, b any real

Assumption on utilities

The second rationality assumption reads:

The agents are able to provide a utility function representing their preferences relations, whenever necessary

Probability issues

The third rationality assumption reads:

The players use consistently the probability laws, in particular they are consistent w.r.t the calculation of expected utilities, they are able to update probabilities according to Bayes rule. . .

For instance, think if you accept to play with me if the game is the following: we toss a coin and if the result is head you take one Euro from me, otherwise you give me one Euro. And ask yourself if play in the case earn 1,000,100 Euro from me when head, and I take 1,000,000 from you if tail.

In this case preferences are not enough, you need utilities and you must calculate the expected value, comparing it with the utility of not playing (getting zero).

Extending decision theory

The next rationality assumption reads:

The player are able to use decision theory, whenever it is possible.

This means that the players are utility maximizers and cost minimizers

Summarizing the rationality assumptions

1. The players are able to consistently rank the outcomes of the game
2. The players are able to provide a utility function for their ranking, whenever necessary
3. The players apply the expected value principle to built their utility functions in presence of random events
4. The players use the apparatus of decision theory anytime it is possible

A further assumption (at least in the foundations of the theory)
The players assume that all players are rational, and that the basic data of the game are common knowledge.
In particular the players are informed about the available strategies of the other players, and about their utility functions.

A first concrete consequence of the axioms

A basic consequence of the decision theory assumption is:

A player does not choose a strategy *a* it she has available a strategy *b* providing her a strictly better result, *no matter what the other players do*

Principle of elimination of strictly dominated strategies.

After an exam, Player one strategy set is $\{18, \dots, 30\}$, player two strategy set is $\{\text{accept}\,, \text{refuse}\,\}$. If player two preference is passing the exam with any grade, rather than repeating it, the action *refuse* is strictly dominated.

Observe, both players do know this. Thus asking for one or two extra points is useless (and would change the game rules. . .)

Finite games

A two player game, where one player, called first, has n strategies and the second m strategies, can be represented by a pair of $n \times m$ matrices, denoted (A, B), often called bimatrix game:

$$\begin{pmatrix} (a_{11}, b_{11}) & \dots & (a_{1j}, b_{1j}) & \dots & (a_{1m}, b_{1m}) \\ \dots & \dots & \dots & \dots & \dots \\ (a_{i1}, b_{i1}) & \dots & (a_{ij}, b_{ij}) & \dots & (a_{im}, b_{im}) \\ \dots & \dots & \dots & \dots & \dots \\ (a_{n1}, b_{n1}) & \dots & (a_{nj}, b_{nj}) & \dots & (a_{nm}, b_{nm}) \end{pmatrix}$$

- $X = \{1, \dots, i, \dots, n\}$, $Y = \{1, \dots, i, \dots, m\}$ are the strategy spaces of the players
- The choices of i and j, respectively, lead to the outcome ij
- The utilities of the players on the outcome ij are, respectively, a_{ij} and b_{ij}.

An example

The two players have two strategies each.

$$\begin{pmatrix} (8,8) & (2,7) \\ (7,2) & (0,0) \end{pmatrix}$$

Utilities of player 1:

$$\begin{pmatrix} 8 & 2 \\ 7 & 0 \end{pmatrix}$$

The second row is strictly dominated by the first, thus player 1 will select the first row.

Another example

$$\begin{pmatrix} (8,8) & (2,7) & (4,10) \\ (7,2) & (0,0) & (3,0) \\ (5,3) & (3,9) & (10,4) \end{pmatrix}$$

- The first player can eliminate ...
- Knowing this the second can eliminate ...
- Knowing this the first can eliminate ...
- The outcome is

Comparing games

The first:

$$\begin{pmatrix} (10,10) & (3,15) \\ (15,3) & (5,5) \end{pmatrix}$$

The second one:

$$\begin{pmatrix} (8,8) & (2,7) \\ (7,2) & (0,0) \end{pmatrix}$$

Observe: in any outcome the players are better off in the first game rather than in the second:

However it is more convenient for them to play the second!

Less is better than more

The first game:

$$\begin{pmatrix} (10,10) & (3,5) \\ (5,3) & (1,1) \end{pmatrix}$$

The second game, containing all possible outcomes the first, and some further outcomes:

$$\begin{pmatrix} (1,1) & (11,0) & (4,0) \\ (0,11) & (10,10) & (3,5) \\ (0,4) & (5,3) & (1,1) \end{pmatrix}$$

Having less available actions can make the players better off!

Definition of non cooperative game

Definition

A two player noncooperative game in strategic form is $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$

X, Y are the strategy sets of the players, f, g their utility functions

Natural extension to many players:

$(X_i, f_i : \times X_i \to \mathbb{R}),\ i = 1, \dots, n$

The Nash equilibrium

A Nash equilibrium profile for the $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$ is a pair $(\bar{x}, \bar{y}) \in X \times Y$ such that:

- $f(\bar{x}, \bar{y}) \geq f(x, \bar{y})$ for all $x \in X$
- $g(\bar{x}, \bar{y}) \geq g(\bar{x}, y)$ for all $y \in Y$

Natural extension to many players:

A strategy profile $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$ such that, for all i:

$$f_i(\bar{x}) \geq f(x_i, \bar{x}_{-i})$$

for all $x_i \in X_i$, where

$(x_i, \bar{x}_{-i})$ is the strategy profile $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \dots, \bar{x}_n)$.

Nash equilibrium and dominated strategies

Suppose $\bar{x}$ is a (weakly) dominant strategy for Pl1:

$f(\bar{x}, y) \geq f(x, y)$ for all x, y.

Then, if $\bar{y}$ maximizes the function (of the variable y, looking at $\bar{x}$ as fixed) $g(\bar{x}, y)$

$(\bar{x}, \bar{y})$ is a NEp

First examples

$$\begin{pmatrix} (3,2) & (1,1) \\ (1,0) & (2,1) \end{pmatrix}$$

Nash equilibria: (first row, first column) payoffs $(3,2)$ and (second row, second column) payoffs $(2,1)$. It is likely that the players will agree on the first.

$$\begin{pmatrix} (3,2) & (0,0) \\ (0,0) & (2,3) \end{pmatrix}$$

Nash equilibria: (first row, first column) payoffs $(3,2)$ and (second row, second column) payoffs $(2,3)$. Note that the players have opposite preferences on the two outcomes

$$\begin{pmatrix} (0,0) & (1,1) \\ (1,1) & (0,0) \end{pmatrix}$$

Nash equilibria: (first row, second column) payoffs $(1,1)$ and (second row, second column) payoffs $(1,1)$. The players are indifferent on the two outcomes, but need coordination to fall in one of them.

Prisoner Dilemma

The most famous example (revisited): a father tells to his two children: do you want that give 1 Euro to you, or that I give 10 Euros to your brother?

$$\begin{pmatrix} (10,10) & (0,11) \\ (11,0) & (1,1) \end{pmatrix}$$

Observe, the unique rational outcome is not only Nash equilibrium, but also obtained with elimination of strictly dominated strategies! This game has exactly the same structure as the prisoner dilemma game.

Tragedy of commons

This game is the two player version of a game known under the name of tragedy of commons

$$\begin{pmatrix} (a,a) & (b,c) \\ (c,b) & (d,d) \end{pmatrix}$$

Relations: $a > d$, $a < c$, $b < d$

Standard situation when two people exploit common resources: it is a strictly doinating strategy to try to use them as much as possible, but since they are finite this makes the situation bad for all.