# Third week

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# Topics



- Finite games with common payoffs
- Payoff equivalence and potential games
  - Existence of equilibria in pure strategies
  - Convergence of best response dynamics
- Examples of potential games
  - Routing games
  - Congestion games
  - Network connection games
  - Location games
- How to find a potential
- Price-of-Anarchy and Price-of-Stability

# Finite games with common payoffs



Consider a finite game with strategy sets $A_i$ and and suppose that all the players have exactly the same payoff $p : A \to \mathbb{R}$:

$$u_i(a_1, \dots, a_n) = p(a_1, \dots, a_n).$$

Take $\bar{a} = (\bar{a}_1, \dots, \bar{a}_n) \in A$ a strategy profile such that $p(\bar{a}) = \max_{a \in A} p(a)$.
Then $\bar{a}$ is clearly a Nash equilibrium in pure strategies.

**Remark**

*There might be other Nash equilibria in pure or mixed strategies.However, playing $\bar{a}$ is the best that every player could ever hope for.*

# Best response dynamics



Consider the following payoff-improving procedure:

1. Start from an arbitrary strategy profile $(a_1, \dots, a_n) \in A$
2. Ask if any player has a better strategy $a_i'$ that strictly increases her payoff
$$u_i(a_i', a_{-i}) > u_i(a_i, a_{-i})$$
   - If yes, replace $a_i$ with $a_i'$ and repeat.
   - Otherwise stop: we have found a pure Nash equilibrium!

Each iteration strictly increases the value $p(a)$ so that no strategy profile $a \in A$ can be visited twice. Since $A$ is a finite set, the procedure must reach a pure Nash equilibrium after at most $|A|$ steps.

**Exercise**

*Prove or disprove that this procedure guarantees to reach the global maximum $\bar{a}$*

# Payoff equivalence



Consider now a general finite game with payoffs $u_i : A \to \mathbb{R}$ not necessarily equal. How do best responses and Nash equilibria change if we add a constant $c_i$ to the payoff of player $i$ ?

$$\tilde{u}_i(a_1, \ldots, a_n) = u_i(a_1, \ldots, a_n) + c_i$$

What if $c_i$ is no longer constant but it depends only on $a_{-i}$ and not on $a_i$?

We say that the payoffs $\tilde{u}_i$ and $u_i$ are difference equivalent for player $i$ if their difference

$$\tilde{u}_i(a_1, \ldots, a_n) - u_i(a_1, \ldots, a_n) = c_i(a_{-i})$$

does not depend on her decision $a_i$ but only on the strategies of the other players.

# Payoff equivalence



By definition, equivalent payoffs are such that for all $a'_i, a_i \in A_i$

$$\tilde{u}_i(a'_i, a_{-i}) - u_i(a'_i, a_{-i}) \cancel{= c_i(a_{-i})} = \tilde{u}_i(a_i, a_{-i}) - u_i(a_i, a_{-i}).$$

Denoting $\Delta f(a'_i, a_i, a_{-i}) = f(a'_i, a_{-i}) - f(a_i, a_{-i})$ this is equivalent to

$$\Delta\tilde{u}_i(a'_i, a_i, a_{-i}) = \Delta u_i(a'_i, a_i, a_{-i}). \quad (1)$$

**Theorem**

*Finite games with equivalent payoffs have the same pure Nash equilibria.*

**Proof** Since $u_i(a_i, a_{-i}) = \tilde{u}_i(a_i, a_{-i}) + c_i(a_{-i})$, it follows that $BR_i(a_{-i}) = \tilde{BR}_i(a_{-i})$ for all $i$ and for all $a_i$. Thus the Nash equilibria profiles are the same in the two cases ($BR_i$ ($\tilde{BR}_i$) denotes the best reaction multifunction of Player $i$ relative to utility $u_i$ ($\tilde{u}_i$))

# Potential games



**Definition**

A finite game with strategy sets $A_i$ and payoffs $u_i : A \to \mathbb{R}$ is called a *potential game* if it is equivalent to a game with identical payoffs: there exists a potential function $p : A \to \mathbb{R}$ such that $p(a) - u_i(a)$ does not depend on $a_i$.

Equivalently: for every $a_{-i} \in A_{-i}$ and all $a_i', a_i \in A_i$ we have

$$\Delta p(a_i', a_i, a_{-i}) = \Delta u_i(a_i', a_i, a_{-i}).$$

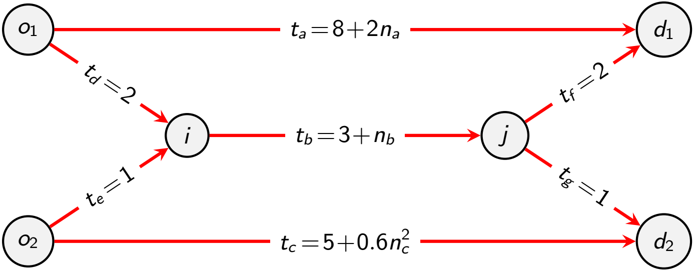
**Corollary**

1. *Every finite potential game has at least one pure Nash equilibrium.*
2. *In a finite potential game every best response iteration reaches a pure Nash equilibrium in finitely many steps.*

# Example 1: Routing games



$n$ drivers travel between different origins and destinations. The transport network is modeled as a graph $(N, A)$ with node set $N$ and arcs $A$. The travel time of an arc $a \in A$ is a non-negative increasing function $t_a = t_a(n_a)$ of the load $n_a = \#$ of drivers using the arc.




Each driver $i$ selects a route $r_i = a_1 a_2 \cdots a_\ell$: a sequence of arcs connecting $o_i \in N$ to $d_i \in N$. Her total travel time is

$$u_i(r_1, \ldots, r_n) = \sum_{a \in r_i} t_a(n_a) \quad ; \quad n_a = \#\{j : a \in r_j\}$$

# Example 1: Routing games



To minimize travel time, drivers may restrict to *simple paths* with no cycles: nodes are visited at most once. Hence, the strategy set for player $i$ is the *finite* set $A_i$ of all simple paths connecting $o_i$ to $d_i$.

**Theorem (Rosenthal'73)**

*A routing game admits the potential*

$$p(r_1,\ldots,r_n)=\sum_{a\in A}\sum_{k=1}^{n_a}t_a(k) \quad ; \quad n_a=\#\{j\colon a\in r_j\}.$$

**Proof** It suffices to note that for $r=(r_1,\ldots,r_n)$ we have

$$p(r)-u_i(r)=\sum_{a\in A}\sum_{k=1}^{n_a}t_a(k)-\sum_{a\in r_i}t_a(n_a)=\sum_{a\in A}\sum_{k=1}^{n_a^{-i}}t_a(k)$$

where $n_a^{-i}=\#\{j\neq i\colon a\in r_j\}$ is the number of drivers other than $i$ using arc $a$. Hence, the difference $p(r)-u_i(r)$ depends only on $r_{-i}$ and not on $r_i$. ∎

# Example 2: Congestion games



A routing game is a special case of the more general class of *Congestion games*. Here each player $i = 1, \ldots, n$ has to perform a certain task which requires some resources taken from a set $R$. The strategy set $A_i$ for player $i$ contains all subsets $a_i \subseteq R$ that allow her to perform the task.

Each resource $r \in R$ has a cost $c_r(n_r)$ which depends on the number of players that use the resource. We do not assume $c_r(\cdot)$ positive or increasing. Player $i$ only pays for the resources she uses

$$u_i(a_1, \ldots, a_n) = \sum_{r \in a_i} c_r(n_r) \quad ; \quad n_r = \#\{j : r \in a_j\}.$$

**Exercise**

*Prove that* $p(a_1, \ldots, a_n) = \sum_{r \in R} \sum_{k=1}^{n_r} c_r(k)$ *is a potential.*

# Example 3: Network connection games



A telecommunication network $(N, A)$ is under construction. Each player $i$ wants a route $r_i$ to be built between a certain origin $o_i$ and a destination $d_i$. The cost $v_a$ of building an arc $a \in A$ is shared evenly among the players who use it.

Hence, the cost for player $i$ is

$$u_i(r_1, \dots, r_n) = \sum_{a \in r_i} \frac{v_a}{n_a} \quad ; \quad n_a = \#\{j \colon a \in r_j\}.$$

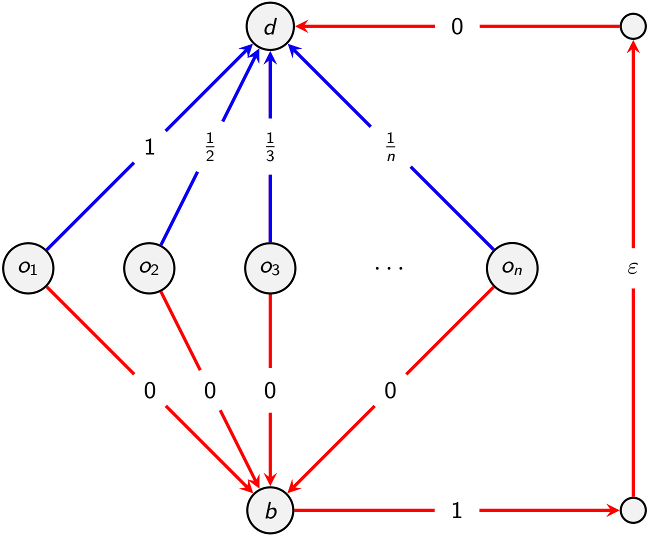
In this case there is an incentive to use congested arcs as this reduces the cost.

This is again a congestion game with potential

$$p(r_1, \dots, r_n) = \sum_{a \in A} v_a \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{n_a}\right).$$

# Example 3: Network connection games





# Example 4: Location games



A group of Internet Service Providers (ISPs) $i = 1, \ldots, n$ compete for providing connectivity to a finite set of customers $k \in K$. Each firm $i$ has to decide where to locate its Data Center, choosing from a finite set of possible sites $A_i$.

Customer $k \in K$ can be served from the different ISP sites $a_i \in A_i$ at a cost $c^k_{a_i}$. Then, firm $i$ will propose to $k$ the competitive price

$$p^k_i(a) = \max\{c^k_{a_i}, \min_{j \neq i} c^k_{a_j}\}.$$

Hence $k$ is served by the ISP with minimal cost and pays the second lowest cost. The profit for firm $i$ is therefore

$$u_i(a_1, \ldots, a_n) = \sum_{k \in K} (p^k_i(a) - c^k_{a_i}).$$

We assume that the value $\pi^k$ that customer $k$ gets from the service is higher that all the costs $c^k_{a_i}$, so that customers are always willing to buy the service.

# Example 4: Location games



**Proposition**

*The location game admits the potential*

$$f(a_1,\dots,a_n)=\sum_{k\in K}[\pi^k-\min_{j=1\dots n} c^k_{a_j}]$$

*which corresponds to the sum of excess utilities for customers and providers.*

**Proof** Considering separately the customers $k$ for which firm $i$ is the minimum cost provider, and the $k$'s for whis it is not, in both cases we get

$$\begin{aligned} f(a)-u_i(a) &= \sum_{k\in K}[\pi^k-\min_{j=1\dots n} c^k_{a_j}-p^k_i(a)+c^k_{a_i}] \\ &= \sum_{k\in K}[\pi^k-\min_{j\neq i} c^k_{a_j}] \end{aligned}$$

where the latter depends only on $a_{-i}$ and not on $a_i$. ■

# How to find a potential



A potential $p : A \to \mathbb{R}$ is characterized by

$$\Delta p(a_i', a_i, a_{-i}) = \Delta u_i(a_i', a_i, a_{-i}).$$

Adding a constant to $p(\cdot)$ changes nothing. Hence fix an arbitrary profile $\bar{a} = (\bar{a}_1, \dots, \bar{a}_n)$ and set $p(\bar{a}) = 0$. Once this is done, the potential $p(\cdot)$ *is determined uniquely*:

$$p(a_1, a_2, \dots, a_n) - p(\bar{a}_1, a_2, \dots, a_n) = u_1(a_1, a_2, \dots, a_n) - u_1(\bar{a}_1, a_2, \dots, a_n)$$
$$p(\bar{a}_1, a_2, \dots, a_n) - p(\bar{a}_1, \bar{a}_2, \dots, a_n) = u_2(\bar{a}_1, a_2, \dots, a_n) - u_2(\bar{a}_1, \bar{a}_2, \dots, a_n)$$
$$\vdots$$
$$p(\bar{a}_1, \bar{a}_2, \dots, a_n) - p(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n) = u_n(\bar{a}_1, \bar{a}_2, \dots, a_n) - u_n(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n)$$

$$\Rightarrow \quad p(a_1, a_2, \dots, a_n) = \sum_{i=1}^{n} [u_i(\bar{a}_1 \dots a_i \dots a_n) - u_i(\bar{a}_1 \dots \bar{a}_i \dots a_n)]$$

# Existence of a potential



Note that instead of changing $a_i \rightsquigarrow \bar{a}_i$ in the order $i = 1, 2, \ldots, n$ we could proceed backwards from $n$ down to 1, or using an arbitrary ordering. After all, using integers to name the players is inmaterial and any order is equally valid.

If the game is potential the sum on the right hand side is *independent of the particular order used*.

The converse is also true. However, checking that all these orders yield the same answer is impractical for more than 2 or 3 players, so this is not of much help.

# Example: computing a potential



Is the following a potential game?

$$\begin{pmatrix} (2,5) & (2,6) & (3,7) & (8,9) & (5,7) \\ (1,4) & (1,5) & (3,7) & (2,3) & (0,2) \\ (6,5) & (2,2) & (2,0) & (6,3) & (3,1) \end{pmatrix}$$

Potential:

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 2 \\ -1 & 0 & 2 & -2 & -3 \\ 4 & 1 & -1 & 2 & 0 \end{pmatrix}$$

# Social cost and efficiency



Nash equilibria need not be Pareto efficient and can be bad for all the players as in the Braess' paradox, the Prisoner's dilemma.
But how *"bad"* can be the outcome of a game ?

To answer this question we must first agree on what is *"good"* and what is *"bad"*. We assume that the quality of a strategy profile $a = (a_1, \dots, a_n)$ is measured through a *social cost* funtion $a \mapsto C(a)$ where $C : A \to \mathbb{R}_+$. The smaller $C(a)$ the better the outcome $a \in A$.
We use as a benchmark the minimal value that a benevolent social planner could achieve

$$Opt = \min_{a \in A} C(a).$$

For $a \in A$ the quotient $\frac{C(a)}{Opt}$ measures how far is $a$ from being optimal.
A large value implies a big loss in social welfare, while a quotient close to 1 implies that $a$ is almost as efficient as an optimal solution.

# Price-of-Anarchy and Price-of-Stability



**Definition**

Let $NE \subseteq A$ be the set of pure Nash equilibria of the game. The Price-of-Anarchy and the Price-of-Stability are defined respectively by

$$PoA = \max_{\bar{a} \in NE} \frac{C(\bar{a})}{Opt} \qquad PoS = \min_{\bar{a} \in NE} \frac{C(\bar{a})}{Opt}$$

Clearly $1 \leq PoS \leq PoA$. Having $PoA \leq \alpha$ means that in every possible pure equilibrium the social cost $C(\bar{a})$ is no worse than $\alpha\, Opt$. When $PoS \leq \alpha$ we can only say that there exists some equilibrium with social cost at most $\alpha\, Opt$.

# Social cost – A typical function

In games where the payoffs $u_i(a)$ represent costs, a natural choice for a social cost is the following function:

$$C(a) = \sum_{i=1}^{n} u_i(a).$$

In this case the individual costs $u_i(a)$ are expressed in some comparable units and scale (monetary, time, weight,...).

**Examples:**

- In the routing game the cost function is the total time traveled by all the players, and can be expressed as

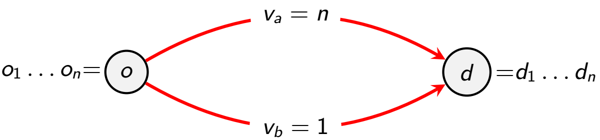
$$C(r_1, \dots, r_n) = \sum_{a \in A} n_a t_a(n_a) \quad ; \quad n_a = \#\{j \colon a \in r_j\}.$$

- In the network connection game the cost function gives the total investment required to connect all the players

$$C(r_1, \dots, r_n) = \sum_{a : n_a > 0} v_a \quad n_a = \#\{j \colon a \in r_j\}.$$

# Example: PoA and PoS — Network connection game





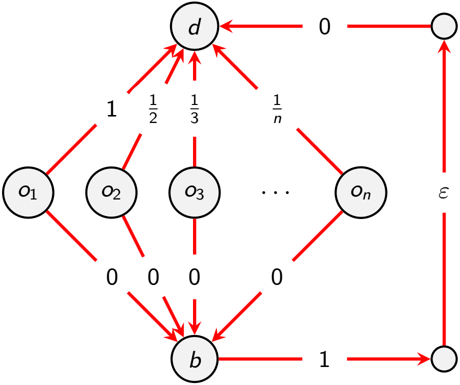

$$Opt = 1$$

$$PoS = 1$$

$$PoA = n \to \infty$$

# Example: PoA and PoS — Network connection game






$$Opt = 1 + \varepsilon$$

$$C(\bar{a}) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = H_n$$

$$PoA = PoS = \frac{H_n}{1+\varepsilon} \sim \ln(n) \to \infty$$

# An estimate for PoS

**Proposition**

*Consider a cost minimization finite potential game with potential $p : A \to \mathbb{R}$, and suppose that the social cost $C : A \to \mathbb{R}_+$ is such that there exist strictly positive constants $\alpha, \beta$ such that*

$$\frac{1}{\alpha} C(a) \leq p(a) \leq \beta C(a) \quad \forall\, a \in A.$$

*Then $PoS \leq \alpha\beta$.*

**Proof** Let $\bar{a}$ be a minimum of $p(\cdot)$ so that $\bar{a}$ is a Nash equilibrium. It follows that for all $a \in A$ we have

$$\frac{1}{\alpha} C(\bar{a}) \leq p(\bar{a}) \leq p(a) \leq \beta C(a)$$

hence $C(\bar{a}) \leq \alpha\beta\, Opt$. ∎

# Application: PoS in network connection games

**Proposition**

*Consider a network congestion game with n players on a general graph $(N, A)$ with arc construction costs $v_a \geq 0$. Then* $PoS \leq H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

**Proof** In this case the potential and the social cost are

$$p(r_1, \ldots, r_n) = \sum_{a \in A} \sum_{k=1}^{n_a} \frac{v_a}{k}$$

$$C(r_1, \ldots, r_n) = \sum_{a : n_a > 0} v_a$$

so that $C(r) \leq p(r) \leq H_n C(r)$ and the previous result yields $PoS \leq H_n$. ∎