Potential Games

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Topics

- Finite games with common payoffs
- Payoff equivalence and potential games
- Existence of equilibria in pure strategies
- Convergence of best response dynamics
- Routing games
- Congestion games
- Network connection games
- Location games
- How to find a potential
- Price-of-Anarchy and Price-of-Stability

Finite games with common payoffs

Consider a finite game with strategy sets X_i and suppose that all the players have the same payoff $p : X \to \mathbb{R}$, that is

$$u_i(x_1, \ldots, x_n) = p(x_1, \ldots, x_n).$$

Take $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in X$ a strategy profile such that $p(\bar{x}) \geq p(x)$ for all strategy profiles $x \in X$.
Then $\bar{x}$ is a Nash equilibrium in pure strategies.

Remark:
There might be other Nash equilibria in pure or mixed strategies.
However, playing $\bar{x}$ is the best that every player could ever hope for.

Best response dynamics

Consider the following payoff-improving procedure:

1. Start from an arbitrary strategy profile $(x_1, \dots, x_n) \in X$
2. Ask if any player has a better strategy x_i' that strictly increases her payoff
$$u_i(x_i', x_{-i}) > u_i(x_i, x_{-i})$$
   - If yes, replace x_i with x_i' and repeat.
   - Otherwise stop: we have found a pure Nash equilibrium profile!

Each iteration strictly increases the value $p(x)$ so that no strategy profile $x \in X$ can be visited twice. Since X is a finite set, the procedure must reach a pure Nash equilibrium after at most $|X|$ steps.

Does this procedure guarantees to reach the global maximum $\bar{x}$?

Payoff equivalence

Consider now a general finite game with payoffs $u_i : X \to \mathbb{R}$. How do best responses and Nash equilibria change if we add a constant c_i to the payoff of player i?

$$\tilde{u}_i(x_1, \dots, x_n) = u_i(x_1, \dots, x_n) + c_i$$

What if c_i is not constant but it depends only on x_{-i} and not on x_i?

Best responses and equilibria remain the same!

The payoffs $\tilde{u}_i$ and u_i are said *diff-equivalent* for player i if the difference

$$\tilde{u}_i(x_1, \dots, x_n) - u_i(x_1, \dots, x_n) = c_i(x_{-i})$$

does not depend on her decision x_i but only on the strategies of the other players.

Payoff equivalence

By definition, diff-equivalent payoffs are such that for all $x_i', x_i \in X_i$

$$\tilde{u}_i(x_i', x_{-i}) - u_i(x_i', x_{-i}) = \tilde{u}_i(x_i, x_{-i}) - u_i(x_i, x_{-i}).$$

Denoting $\Delta f(x_i', x_i, x_{-i}) = f(x_i', x_{-i}) - f(x_i, x_{-i})$ this can be rewritten as

$$\Delta \tilde{u}_i(x_i', x_i, x_{-i}) = \Delta u_i(x_i', x_i, x_{-i}). \quad (1)$$

Theorem

Finite games with diff-equivalent payoffs have the same pure Nash equilibria.

Proof: A profile $(x_1, \dots, x_n)$ is a pure Nash equilibrium iff the payoff increments when moving from x_i to any other x_i' are non-positive $\Delta u_i(x_i', x_i, x_{-i}) \leq 0$. It follows from (1) that pure Nash equilibria are the same for u_i and $\tilde{u}_i$. ∎

Prove that this result also holds for mixed equilibria

Potential games

Definition

A finite game with strategy sets X_i and payoffs $u_i : X \to \mathbb{R}$ is called a potential game if it is diff-equivalent to a game with common payoffs, that is, there exists a potential function $p : X \to \mathbb{R}$ such that for each i, for every $x_{-i} \in X_{-i}$, and all $x_i', x_i \in X_i$ we have

$$\Delta u_i(x_i', x_i, x_{-i}) = \Delta p(x_i', x_i, x_{-i}).$$

Corollary

1. *Every finite potential game has at least one pure Nash equilibrium.*
2. *In a finite potential game every best response iteration reaches a pure Nash equilibrium in finitely many steps.*

A toy example

$$\begin{pmatrix} (10,10) & (0,11) \\ (11,0) & (1,1) \end{pmatrix}$$

A potential

$$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

For Player 2

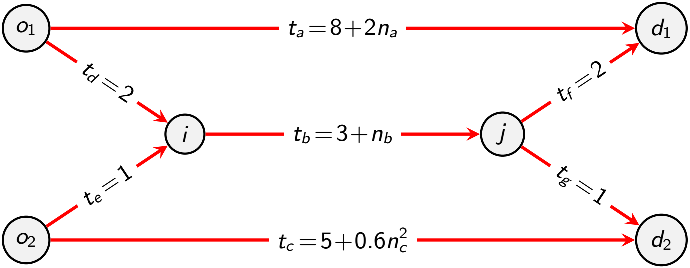
- Differences when the first row is fixed: $11-10=1-0$
- Differences when the second row is fixed: $1-0=2-1$

For Player 1

- Differences when the first column is fixed: $11-10=1-0$
- Differences when the second column is fixed: $1-0=2-1$

Example 1: Routing games

Consider n drivers traveling between different origins and destinations in a city. The transport network is modeled as a graph (N, A) with node set N and arcs A. Because of congestion, the travel time of an arc $a \in A$ is a non-negative increasing function $t_a = t_a(n_a)$ of the load $n_a = \#$ of drivers using the arc. We set $t_a(0) = 0$.



One pure strategy for i is a route $r_i = a_1 a_2 \cdots a_\ell$, that is, a sequence of arcs connecting her origin $o_i \in N$ to her destination $d_i \in N$. Her total travel time is

$$u_i(r_1, \dots, r_n) = \sum_{a \in r_i} t_a(n_a) \quad ; \quad n_a = \#\{j : a \in r_j\}$$

Example 1: Routing games

To minimize travel time, drivers may restrict to simple paths with no cycles: nodes are visited at most once. Hence, the strategy set for player i is the set X_i of all simple paths connecting o_i to d_i.

Theorem (Rosenthal'73)

A routing game admits the potential

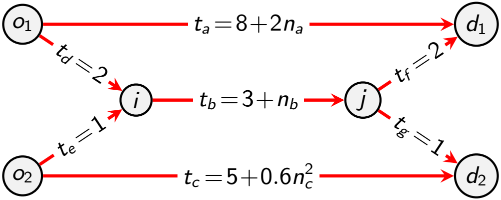
$$p(r_1, \ldots, r_n) = \sum_{a \in A} \sum_{k=0}^{n_a} t_a(k) \quad ; \quad n_a = \#\{j \colon a \in r_j\}.$$

Proof It suffices to note that for $r = (r_1, \ldots, r_n)$ we have

$$p(r) - u_i(r) = \sum_{a \in A} \sum_{k=0}^{n_a} t_a(k) - \sum_{a \in r_i} t_a(n_a) = \sum_{a \in A} \sum_{k=1}^{n_a^{-i}} t_a(k)$$

where $n_a^{-i} = \#\{j \neq i \colon a \in r_j\}$ is the number of drivers other than i using arc a. Hence, the difference $p(r) - u_i(r)$ depends only on r_{-i} and not on r_i. ■

Example revisited



Two players go from O_1 to d_1 and one from O_2 to d_2. $r_1 = a$, $r_2 = dbf$, $r_3 = ebg$.
$\sum_{k=1}^{n_a} t_a(k)$ for every arc, under the profile r:

1. a 10
2. b $4 + 5$
3. c 0
4. d 2
5. e 1
6. f 2
7. g 1

Costs:

1. for player $1 = 10$ (arc a)
2. for player $2 = 2$ (arc d)$+$ 5 (arc b)$+$ 2 (arc f)
3. for player $3 = 1$ (arc d)$+$ 5 (arc b)$+$ 1 (arc g)

Difference $p(r_1, r_2, r_3) - u_1(r_1, r_2, r_3)$ depends only from r_2, r_3 and the same for the other players.

Example 2: Congestion games

A routing game is a special case of the more general class of *Congestion games*. Here each player $i = 1, \dots, n$ has to perform a certain task which requires some resources taken from a set R. The strategy set X_i for player i contains all subsets $x_i \subseteq R$ that allow her to perform the task.

Each resource $r \in R$ has a cost $c_r(n_r)$ which depends on the number of players that use the resource. Player i only pays for the resources she uses

$$u_i(x_1, \dots, x_n) = \sum_{r \in x_i} c_r(n_r) \quad ; \quad n_r = \#\{j : r \in x_j\}.$$

Verify that $p(x_1, \dots, x_n) = \sum_{r \in R} \sum_{k=1}^{n_r} c_r(k)$ is a potential.

Observe: here u_i represents a cost for Player i

Example 3: Network connection games

A telecommunication network (N, A) is under construction. Each player i wants a route r_i to be built between a certain origin o_i and a destination d_i. The cost v_a of building an arc $a \in A$ is shared evenly among the players who use it.

Hence, the cost for player i is

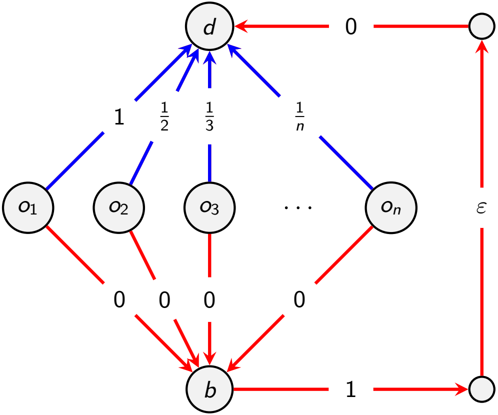
$$u_i(r_1, \dots, r_n) = \sum_{a \in r_i} \frac{v_a}{n_a} \quad ; \quad n_a = \#\{j : a \in r_j\}.$$

In this case there is an incentive to use congested arcs as this reduces the cost.

This is again a congestion game with potential

$$p(r_1, \dots, r_n) = \sum_{a \in A : n_a > 0} v_a\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{n_a}\right).$$

Example 3: Network connection games



Example 4: Location games

A group of Internet Service Providers (ISPs) $i = 1, \ldots, n$ compete for providing connectivity to a finite set of customers $k \in K$. Each firm i has to decide where to locate its Data Center, choosing from a finite set of possible sites X_i.

Customer $k \in K$ can be served from the different ISP sites $x_i \in A_i$ at a cost $c^k_{x_i}$. Then, firm i will propose to k the competitive price

$$p^k_i(x) = \max\{c^k_{x_i}, \min_{j \neq i} c^k_{x_j}\}.$$

Hence k is served by the ISP with minimal cost and pays the second lowest cost. The profit for firm i is therefore

$$u_i(x_1, \ldots, x_n) = \sum_{k \in K} (p^k_i(x) - c^k_{x_i}).$$

We assume that the value π^k that customer k gets from the service is higher that all the costs $c^k_{a_i}$, so that customers are always willing to buy the service.

Example 4: Location games

Proposition

The location game admits the potential

$$p(x_1, \ldots, x_n) = \sum_{k \in K} [\pi^k - \min_{j=1\ldots n} c^k_{x_j}]$$

which corresponds to the sum of excess utilities for customers and providers.

Proof Considering separately the customers k for which firm i is the minimum cost provider, and the k's for which it is not, in both cases we get

$$\begin{aligned} f(x) - u_i(x) &= \sum_{k \in K} [\pi^k - \min_{j=1\ldots n} c^k_{x_j} - p^k_i(x) + c^k_{x_i}] \\ &= \sum_{k \in K} [\pi^k - \min_{j \neq i} c^k_{x_j}] \end{aligned}$$

where the latter depends only on x_{-i} and not on x_i. ■

How to find a potential

A potential $p : X \to \mathbb{R}$ is characterized by

$$\Delta p(x_i', x_i, x_{-i}) = \Delta u_i(x_i', x_i, x_{-i}).$$

Adding a constant to $p(\cdot)$ provides a new potential.
Fix an arbitrary profile $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$ and set $p(\bar{x}) = 0$.
Now the potential $p(\cdot)$ is determined uniquely:

$$\begin{array}{rcl}
p(x_1, x_2, \dots, x_n) - p(\bar{x}_1, x_2, \dots, x_n) & = & u_1(x_1, x_2, \dots, x_n) - u_1(\bar{x}_1, x_2, \dots, x_n) \\
p(\bar{x}_1, x_2, \dots, x_n) - p(\bar{x}_1, \bar{x}_2, \dots, x_n) & = & u_2(\bar{x}_1, x_2, \dots, x_n) - u_2(\bar{x}_1, \bar{x}_2, \dots, x_n) \\
& \vdots & \\
p(\bar{x}_1, \bar{x}_2, \dots, x_n) - p(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) & = & u_n(\bar{x}_1, \bar{x}_2, \dots, x_n) - u_n(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) \\
\hline
\end{array}$$

$$\Rightarrow \quad p(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} [u_i(\bar{x}_1 \dots x_i \dots x_n) - u_i(\bar{x}_1 \dots \bar{x}_i \dots x_n)]$$

Existence of a potential

If the game admits a potential the sum on the right hand side of the previous slide is independent of the particular order used.

The converse is also true. However, checking that all these orders yield the same answer is impractical for more than 2 or 3 players.

Example: computing a potential

Is the following a potential game?

$$\begin{pmatrix} (2,5) & (2,6) & (3,7) & (8,9) & (5,7) \\ (1,4) & (1,5) & (3,7) & (2,3) & (0,2) \\ (6,5) & (2,2) & (0,0) & (6,3) & (3,1) \end{pmatrix}$$

Potential:

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 2 \\ -1 & 0 & 2 & -2 & -3 \\ 4 & 1 & -1 & 2 & 0 \end{pmatrix}$$

Social cost and efficiency

Nash equilibria need not be Pareto efficient and can be bad for all the players as in the Braess' paradox, the Prisoner's dilemma, or the Tragedy of the commons.
An important question is to quantify how bad can be the outcome of a game.
To answer this question it is necessary to define what is good and what is bad.
Different choices are possible. We assume from now on that, like in most previous examples, costs, rather than utilities, of the players are given.
The quality of a strategy profile $x = (x_1, \dots, x_n)$ is measured through a social cost function $x \mapsto C(x)$ where $C : X \to \mathbb{R}_+$. The smaller $C(x)$ the better the outcome $x \in X$. The benchmark is the minimal value that a benevolent social planner could achieve

$$Opt = \min_{x \in X} C(x).$$

For $x \in X$ the quotient $\frac{C(x)}{Opt}$ measures how far is x from being optimal. A large value implies a big loss in social welfare, a quotient close to 1 implies that x is almost as efficient as an optimal solution.

Price-of-Anarchy and Price-of-Stability

Definition

Let $NE \subseteq X$ be the set of pure Nash equilibria of the game. The Price-of-Anarchy and the Price-of-Stability are defined respectively by

$$PoA = \max_{\bar{x} \in NE} \frac{C(\bar{x})}{Opt} \qquad ; \qquad PoS = \min_{\bar{x} \in NE} \frac{C(\bar{x})}{Opt}$$

$1 \leq PoS \leq PoA$

- $PoA \leq \alpha$ means that in every possible pure equilibrium the social cost $C(\bar{x})$ is no worse than $\alpha \, Opt$
- $PoS \leq \alpha$ means that there exists some equilibrium with social cost at most $\alpha \, Opt$.

Social cost – Egalitarian function

A natural cost function aggregates the costs of all the players

$$C(a) = \sum_{i=1}^{n} u_i(a)$$

Example

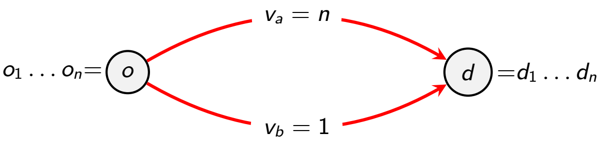
- *In the routing game the egalitarian function is the total time traveled by all the players*

$$C(r_1, \dots, r_n) = \textstyle\sum_{x \in X} n_x t_x(n_x) \quad ; \quad n_x = \#\{j : x \in r_j\}.$$

- *In the network connection game the egalitarian function gives the total investment required to connect all the players*

$$C(r_1, \dots, r_n) = \textstyle\sum_{x \in X : n_x > 0} v_x \quad ; \quad n_x = \#\{j : x \in r_j\}.$$

Example: PoA and PoS — Network connection game

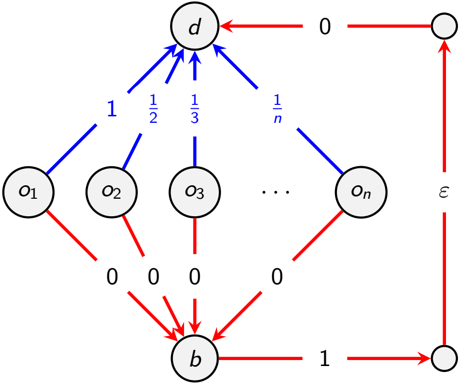


$$Opt = 1$$

$$PoS = 1$$

$$PoA = n \to \infty$$

Example: PoA and PoS — Network connection game



$$Opt = 1 + \varepsilon$$

$$C(\bar{x}) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = H_n$$

$$PoA = PoS = \frac{H_n}{1+\varepsilon} \sim \ln(n) \to \infty$$

Verify that the unique Nash equilibrium profile prescribing to each player to connect directly to the destination can be obtained by elimination of strictly dominated strategies

An estimate for PoS

Proposition

Consider a cost minimization finite potential game with potential $p : X \to \mathbb{R}$, and suppose that there exist $\alpha, \beta > 0$ such that

$$\frac{1}{\alpha} C(x) \leq p(x) \leq \beta C(x) \quad \forall\, x \in X.$$

Then PoS $\leq \alpha\beta$.

Proof Let $\bar{x}$ be a minimum of $p(\cdot)$ so that $\bar{x}$ is a Nash equilibrium. For all $x \in X$

$$\frac{1}{\alpha} C(\bar{x}) \leq p(\bar{x}) \leq p(x) \leq \beta C(x)$$

Since this is true for all x, then $C(\bar{x}) \leq \alpha\beta\, Opt$. ∎

Application: PoS in network connection games

Proposition

Consider a network congestion game with n players on a general graph (N, X) with arc construction costs $v_x \geq 0$. Then $PoS \leq H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

Proof In this case the potential and the social cost are

$$\begin{aligned} p(r_1, \ldots, r_n) &= \sum_{x \in X} \sum_{k=1}^{n_x} \frac{v_x}{k} \\ C(r_1, \ldots, r_n) &= \sum_{x \in X : n_x > 0} v_x \end{aligned}$$

so that $C(r) \leq p(r) \leq H_n C(r)$ and the previous result yields $PoS \leq H_n$.
■

A final remark

In case a game deals with utilities rather than costs, one defines

$$Opt = \max_{x \in X} U(x).$$

Definition

Let $NE \subseteq X$ be the set of pure Nash equilibria of the game. The Price-of-Anarchy and the Price-of-Stability for a utility game are defined respectively by

$$PoA = \max_{\bar{x} \in NE} \frac{Opt}{U(\bar{x})} = \frac{Opt}{\min_{\bar{x} \in NE}(U(\bar{x}))} \quad PoS = \min_{\bar{x} \in NE} \frac{Opt}{U(\bar{x})} = \frac{Opt}{\max_{\bar{x} \in NE}(U(\bar{x}))}$$

This is to have that high PoS and PoA continue to indicate games with bad behavior of Nash equilibrium profiles.