The Nash bargaining model

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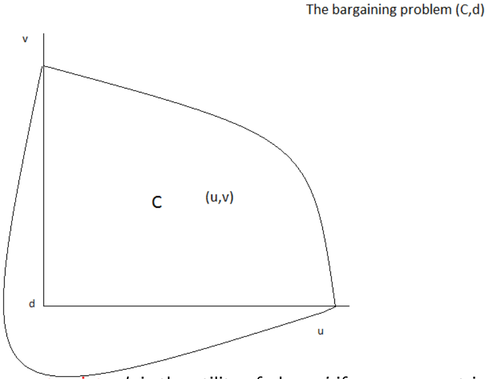
Definition of bargaining problem



d is the disagreement point: d_i is the utility of player i if an agreement is not reached
C is the set of all possible (utility) outcomes: $(u, v) \in C$ means that a possible outcome of the bargaining process assigns utility u (v) to player 1 (2)
It can be seen as a cooperative game (NTU) with two players

The set of the bargaining problems

$\mathcal{C} = \{(C, d)\}$ such that

- C is closed bounded convex subset of $\mathbb{R}^2$
- $d \in \mathbb{R}^2$
- there exists $x \in C : x_1 > d_1, x_2 > d_2$

Definition

A solution for the bargaining problem is a function

$$f : \mathcal{C} \to \mathbb{R}^2$$

such that $f[(C, d)] \in C$*, for all* $(C, d) \in \mathcal{C}$

- C closed bounded is no restrictive assumption
- Convexity is more delicate but acceptable
- Assumption on x means that both players have interest in bargaining

Properties for a solution

The following are interesting properties for f:

- Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is the following transformation of the plane: $L(x_1, x_2) = (ax_1 + c, bx_2 + e)$, with $a, b > 0$ and $c, e \in \mathbb{R}$. Then
$$f[L(C), L(d)] = L[f(C, d)]$$
- Suppose $S : \mathbb{R}^2 \to \mathbb{R}^2$ is the following transformation of the plane: $S(x_1, x_2) = (x_2, x_1)$. Suppose moreover a game (C, d) fulfills $(S(C), S(d)) = (C, d)$. Then
$$f(C, d) = S[f(C, d)]$$
- Given the two problems (A, d) and (C, d)if
$$A \supset C \wedge f[(A, d)] \in C$$
then $f[(C, d)] = f[(A, d)]$
- Given (C, d),
$$y \in C \wedge u \in C : u_1 > y_1, u_2 > y_2$$
implies $f[(C, x)] \neq y$

Meaning

The properties are called

- Invariance with respect to admissible transformations of utility functions
- Symmetry. In a problem (C, d) fulfilling $(S(C), S(d)) = (C, d)$ the players are symmetric
- Independence from irrelevant alternatives, for short IIA
- Efficiency

Remark

The function L providing admissible transformation of utility functions is invertible: $L^{-1}(y_1, y_2) = (\frac{y_1}{a} - \frac{c}{a}, \frac{y_2}{b} - \frac{d}{b})$ represents an admissible transformation of utility functions as well

The Nash bargaining theorem

Theorem

There is one and only one f satisfying the above properties. Precisely, if $(C, d) \in \mathcal{C}$, $f[(C, d)]$ is the point maximizing the function

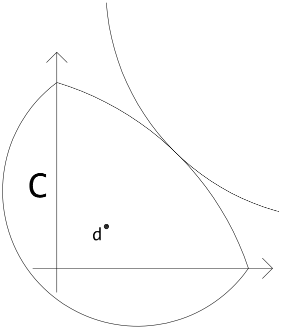
$$g(u, v) = (u - d_1)(v - d_2)$$

on the set

$$C \cap \{(u, v) : u \geq d_1, v \geq d_2\}$$

In other words, players must maximize the product of their utilities over the set of the interesting outcomes

The solution graphically



The proof

Proof Outline.

- f is well defined: the point maximizing g on C exists, since g is a continuous function and the domain C is closed convex bounded. Uniqueness of the maximum point is provided by strict quasi concavity of the function g.
- The verification that C satisfies the other properties is not difficult. In particular IIA is trivial, and efficiency is straightforward
- Uniqueness: call h a function fulfilling the properties. Symmetry and efficiency imply $h = f$ on the subclass of the symmetric games. Now take a general problem (C, d) and, by means of the property of invariance with respect to admissible transformation of utilities send d to the origin and the point $f(C, d)$ to $(1, 1)$
  $(L(x_1, x_2) = (\frac{1}{f_1[(C,d)]-d_1}x_1 - \frac{d_1}{f_1[(C,d)]-d_1}, \frac{1}{f_2[(C,d)]-d_2}x_2 - \frac{d_2}{f_2[(C,d)]-d_2})$. Then

  $$L(C) \subset A = \{(u, v) : \ u, v \geq 0, \ u + v \leq 2\}$$

  $(A, 0)$ is a symmetric game, so that $f(A, 0) = h(A, 0) = (1, 1)$. The independence of irrelevant alternatives provides $h(L(C), 0) = (1, 1) = f(L(C), 0)$. Apply again the property of invariance with respect to admissible transformation of utilities to go back to the original bargaining situation, and conclude from this.

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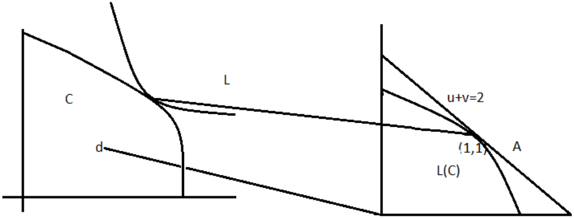
Picture for uniqueness



The transformation L sends d to $(0,0)$ and the Nash solution to $(1,1)$. Apply IIA to $(L(C),0)$ and $(A,(0,0)$ to conclude that $h[(L(C),(0,0)] = f[(L(C),(0,0)]$ and go back with the inverse of L

An interesting fact (1)

The problem is dividing a pie of 1, Player one will get x Player 2 $1-x$ with utilities $u_1(x), u_2(1-x)$ with u_i is increasing, concave and twice differentiable such that $u_i(0) = 0$

x must maximize $g(z) = u_1(z)u_2(1-z)$

It must be $g'(x) = 0$. Thus the equation:

$$\frac{u_1'(x)}{u_1(x)} = \frac{u_2'(1-x)}{u_2(1-x)}$$

must hold

The two curves $\frac{u_1'(z)}{u_1(z)}$ and $\frac{u_2'(1-z)}{u_2(1-z)}$ intersect at the unique point with abscissa x

An interesting fact (2)

Suppose the second player changes his utility function from u_2 to $h \circ u_2$, h as u_i, call y the new quantity assigned to Player 1

The above equation becomes:

$$\frac{u_1'(y)}{u_1(y)} = \frac{h'(u_2(1-y))u_2'(1-y)}{h(u_2(1-y))}$$

Since for every z

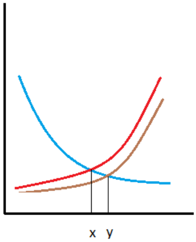
$$\frac{u_2'(1-z)}{u_2(1-z)} \geq \frac{h'(u_2(1-z))u_2'(1-z)}{h(u_2(1-z))}$$

it follows $y > x$

Applying h to u_2 means that the second player becomes more risk averse

Thus according to Nash the more risk averse one player is, the less he get: a well known fact in experimental economics.

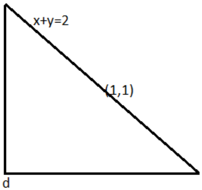
An interesting fact: the picture

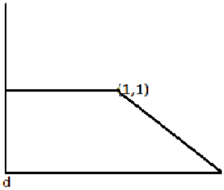


$$\frac{u_1'(z)}{u_1(z)} \quad \frac{u_2'(1-z)}{u_2(1-z)} \quad \frac{h'(u_2(1-z))u_2'(1-x)}{h(u_2(1-x))}$$

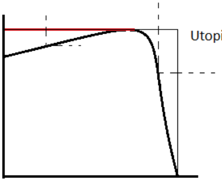
How realistic is the model?

- The least realistic assumption: player's utilities are common knowledge
- Convexity is a bit restrictive
- Uniqueness is based on the fact that the domain of the function is quite large
- The IIA assumption can be criticized





Alternative assumption



$$g_C(x) = \begin{cases} y & \text{if } (x,y) + \mathbb{R}^2_+ \cap C = (x,y) \\ U_2 & \text{otherwise} \end{cases}$$

$U = (U_1, U_2) :=$ Utopia point, where $U_i = \max u_i$ on $C \cap \{(u_1, u_2) : u_1 \geq d_1, u_2 \geq d_2\}$

Monotonicity assumption

Definition

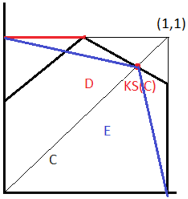
Let $f : \mathcal{C} \to \mathbb{R}^2$ be a solution of the bargaining problem. Then f satisfies the monotonicity assumption for player 1 if for every pair of problems (C, d), $(\hat{C}, d)$ such that $U_1[(C, d)] = U_1[(\hat{C}, d)]$ and $g_C \leq g_{\hat{C}}$, it holds that $f_2[(\hat{C}, d)] \geq f_2[(C, d)]$

Theorem

There is one and only one solution f fulfilling efficiency, invariance with respect to admissible transformation of utilities, symmetry and monotonicity for both players: f associates to every (C, d) the efficient point lying on the line joining the points d and U

f is called the Kalai-Smorodinski solution

Proof of the KS theorem



Let f be any function fulfilling the axioms. In the picture

- All problems have $(0,0)$ as disagreement point, and $(1,1)$ as utopia point
- The problem E is symmetric
- In every symmetric problem KS and f must coincide: $f(E) = KS(E)$
- by monotonicity $f(E) = f(D) = f(C)$, $KS(E) = KS(D) = KS(C)$

$\therefore$

$f(C) = KS(C)$. By invariance with respect to admissible transformation of utilities it is $f[(C,d)] = KS[(C,d)]$ for all (C,d) ■