Matching Problems

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Background setting

Problems introduced in 1962 by Gale and Shapley for the study of two sided markets:

1) workers & employers
2) interns & hospitals
3) students & universities
4) women & men
5) pairs of donor patients in a kidney transplant program
6) $\cdots$

First basic question: what does mean find a (efficient) solution in this case?

First basic version

WOMEN & MEN

Basic ingredients

- Two groups: $\mathcal{W}$ and $\mathcal{M}$
- Each element of $\mathcal{W}$ ($\mathcal{M}$) has a ranking on the elements of $\mathcal{M}$ ($\mathcal{W}$)
- Goal: forming efficient pairs

This is called one to one matching

Further assumptions

- The two groups have the same number of elements
- All elements rank all elements of the other group

Typical example: matching women and men, with the same number of women and men, and better to be matched in any case than being alone

More definitions

Definition

A *matching problem* *is given by:*

1) *a pair of sets* $\mathcal{W}$ *and* $\mathcal{M}$ *with the same cardinality*
2) *a pair preference profiles* $(\{\succsim_w\}_{w\in\mathcal{W}}, \{\succsim_m\}_{m\in\mathcal{M}})$, *with* $\succsim_m$ *defined on* $\mathcal{W}$ *and conversely* $\succsim_w$ *defined on* $\mathcal{M}$

Definition

A *matching* *is a bijection* $b : \mathcal{W} \to \mathcal{M}$

An example

From now on men small letters in blue, women capital letters in red

$$\mathcal{W} = \{\text{Anna}, \text{Giulia}, \text{Maria}\}$$

$$\mathcal{M} = \{\text{Bob}, \text{Frank}, \text{Emanuele}\}$$

The following is a matching

$$[(\text{Anna}, \text{Emanuele}), (\text{Giulia}, \text{Bob}), (\text{Maria}, \text{Frank})] .$$

Stable matching

Definition

A pair $W - -m$ objects to the matching Λ if m and W both prefer each other to the person paired to them in the matching Λ.

For instance:

$$\Lambda = \{(m, Z), (b, W), \dots\}$$

and

$$m \succ_W b \qquad \wedge \qquad W \succ_m Z.$$

Definition

A matching Λ is called stable provided there is no pair woman–man objecting to Λ

Example

$$\mathcal{M} = \{a, b, c, d\}, \mathcal{W} = \{A, B, C, D\}$$

Preferences men:

- $D \succ_a C \succ_a B \succ_a A$, $B \succ_b A \succ_b C \succ_b D$
- $D \succ_c A \succ_c C \succ_c B$, $A \succ_d D \succ_d C \succ_d B$.

Preferences women:

- $b \succ_A a \succ_A c \succ_A d$, $b \succ_B d \succ_B a \succ_B c$
- $b \succ_C d \succ_C c \succ_C a$, $b \succ_D c \succ_D d \succ_D a$.

Two matchings:

$$\Lambda = \{(a, A), (b, C), (c, B), (d, D)\} \ \Omega = \{(a, C), (b, B), (c, D), (d, A)\}$$

Λ unstable a–B $\qquad$ Ω stable

Existence of stable matching

Theorem

Every matching problem admits a stable matching

Proof

Constructive proof: algorithm night by night

1) Stage 1a) Every woman visits, the first night, her most preferred choice. Stage 1b) Every man chooses among the women he finds in front of his house (if any). If every woman is matched, the algorithm ends, otherwise go to stage 2
2) Stage 2a) Every woman dismissed at the previous stage visits her second choice. Stage 2b) Every man chooses among the women he finds in front of his house (if any), he dismisses the woman already in the house, if someone at the door is better for him. If every woman is matched, the algorithm ends, otherwise go to stage k
3) Stage ka) Every woman dismissed at the previous stage visits her choice after the man dismissing her. Stage kb) Every man chooses among the women he finds in front of his house (if any), he dismisses the woman already in the house, if someone at the door is better for him. If every woman is matched, the algorithm ends, otherwise go to stage k.

Claim. The algorithm ends and the resulting matching is stable

Preliminary remarks

Easy to see

- The women go down with their preferences along the algorithm
- The men go up with their preferences along the algorithm
- If a man is visited at stage r, then from stage $r+1$ on he will never be alone
- The algorithm generically provides two matchings

Proof: continued

Fact 1 The algorithm ends, and every man is matched to a woman. Actually every woman can at most visit n men. Thus a first estimate is that at most n^2 days are needed. An immediate better estimate is available noticing that the first night all women are involved, thus $n^2 - n + 1$ is a better estimate (yet, it is not sharp). Moreover, every man is visited at some stage since every woman likes better to be paired than to remain alone

Fact 2 No woman can be part of an objecting pair for the resulting matching
Consider the woman W

- She cannot be part of an objecting pair with a man m she did not visit: she is married to a man preferred to all of them
- She cannot be part of an objecting pair with a man m she already visited either: dismissed in favor of another woman, by transitivity she cannot be preferred to the woman matched to m

The matching built by the algorithm is stable ■

Example revisited

Preferences men:

- $D \succ_a C \succ_a B \succ_a A$, $\quad B \succ_b A \succ_b C \succ_b D$
- $D \succ_c A \succ_c C \succ_c B$, $\quad A \succ_d D \succ_d C \succ_d B$.

Preferences women:

- $b \succ_A a \succ_A c \succ_A d$, $\quad b \succ_B d \succ_B a \succ_B c$
- $b \succ_C d \succ_C c \succ_C a$, $\quad b \succ_D c \succ_D d \succ_D a$

Men visiting:

$$\{(a, C), (b, B), (c, D), (d, A)\}$$

Women visiting

$$\{(a, A), (b, B), (c, D), (d, C)\}$$

One more example

Preferences men

- $A \succ_a B \succ_a C, \quad C \succ_b A \succ_b B, \quad B \succ_c A \succ_c C$

Preferences women

- $c \succ_A a \succ_A b, \quad b \succ_B a \succ_B c, \quad a \succ_C b \succ_C c$

Men visiting

$$\{(a, A), (b, C), (c, B)\}$$

Women visiting

$$\{(c, A), (b, B), (a, C)\}$$

One more?

$$\{(a, B), (b, C), (c, A)\}$$

Some numbers

$|(\text{matching})| = n!$

$|(\text{stable matching})| = ?$

Usually not many!

However there is a method allowing to built many stable matching

For instance:

$n = 8$ 269, $n = 16$ 195472 $n = 32$ 104310534400

Comparing matchings

Let us consider two matchings Δ and Θ

Definition

Write $\Delta \succeq_m \Theta$ *if for every man*

- *either he is associated to the same woman in the two matchings*
- *or he is associated to a preferred woman in* Δ *than in* Θ

Write $\Delta \succeq_w \Theta$ *interchanging men and women in the above*

Remark

Easy to see

- $\Delta \succeq_m (\succeq_w)\Delta$ *reflexivity*
- $\Delta \succeq_m (\succeq_w)\Theta \wedge \Theta \succeq_m (\succeq_w)\Lambda \Rightarrow \Delta \succeq_m (\succeq_w)\Lambda$ *transitivity*
- *the two preorder relations are not complete*

Women versus men

Theorem

Let Δ and Θ be stable matchings. Then $\Delta \succeq_m \Theta$ if and only if $\Theta \succeq_w \Delta$

Suppose $\Delta \succeq_m \Theta$, and $(a, A) \in \Delta$, $(b, A) \in \Theta$. Must show

$$b \succ_A a$$

Suppose in $(a, F) \in \Theta$. Since $\Delta \succeq_m \Theta$

$$A \succ_a F$$

Moreover

$$\{(a, F), (b, A)\} \subset \Theta$$

$\therefore$

$$b \succ_A a$$

Ordering stable matching

Theorem

Let $\Lambda_m(\Lambda_w)$ be the men (women) visiting matching and let Θ be another stable matching. Then

$$\Lambda_m \succeq_m \Theta \succeq_m \Lambda_w \qquad \Lambda_w \succeq_w \Theta \succeq_w \Lambda_m$$

Women visiting is the best for the women!

The proof

Proving that a woman cannot be rejected by a man available to her, by induction on the days of visit

First day. Suppose A is rejected by a in favor of B and that there is a stable set Δ such that

$$\{(a, A), (b, B)\} \subset \Delta$$

Then, since

$$B \succ_a A$$

then

$$b \succ_B a$$

Impossible. By assumption B is visiting a the first day, thus a is her preferred man!

The proof continued

Suppose no woman was rejected by an available man the days $1, \dots, k-1$. See that no woman can be refused by an available man the day k. Using the same argument as before

By contradiction, suppose A is rejected by a in favor of B in day k and that there is a stable set Δ such that

$$\{(a, A), (b, B\} \subset \Delta$$

Then, since

$$B \succ_a A$$

implying

$$b \succ_B a$$

Since B is visiting a, but likes better b, then B visited b some day before and was rejected, against the inductive assumption

Proof finished

We must prove that for every stable matching Θ it is

$$\Lambda_w \succeq_w \Theta$$

Let $(a, A) \in \Lambda_w$. Must show that if $(b, A) \in \Theta$ then

$$a \succ_A b$$

Suppose the contrary. Then A visited b before a and was rejected. Impossible!

To conclude, use previous theorem and symmetry between men and women.

Further result

Definition

Let Δ, Θ be two matching. Define a new matching $\Delta \vee_w \Theta$ as the matching where each woman is paired to the preferred man between the two paired to her in Δ and Θ

Theorem

Let Δ, Θ be two stable matching. Then $\Delta \vee_w \Theta$ is stable

Mathematically, $\vee_w$ provides a lattice structure to the set of stable matchings

A key issue

Is it possible that people lie on their true preferences?

Consider the case women visiting

- It can be shown that women do not have incentive to lie
- Examples show for a man could be convenient to lie

An extension

Suppose men are more and women and men like better to be paired than staying alone

Definition

A matching is a function assigning to every man an element of the set $\mathcal{W} \cup \{single\}$, with every woman associated to a man

Definition

A pair $A - a$ objects to a matching if

- *a is either single or paired to a woman for him worse than A*
- *A loves more a than the man paired to her in the matching*

A matching is stable if there are no objecting pairs

Theorem

A stable matching always exists. The set of the men which are not married remains the same all stable matchings

Further extensions

1) Getting married, but not at any cost
2) Polygamous matching
3) Unisexual matching

1) adaptation of the idea of stable matching, the algorithm works
2) adaptation of the idea of stable matching, the algorithm works
3) a stable matching need not to exist