The Nucleolus

Roberto Lucchetti

Politecnico di Milano

Summary of the slides

1. The excess
2. The lexicographic order
3. The Nucleolus
4. The nucleolus and the core

Excess

A TU game v is given

Definition

The excess of a coalition A over the imputation x is

$$e(A,x) = v(A) - \sum_{i \in A} x_i$$

$e(A,x)$ is a measure of the dissatisfaction of the coalition A with respect to the assignment of the imputation x

Remark

An imputation x of the game v belongs to $C(v)$ if and only if $e(A,x) \leq 0$ for all A

Definition

The lexicographic vector attached to the imputation x is the $(2^n-1)-th$ dimensional vector $\theta(x)$ such that

1. $\theta_i(x) = e(A, x)$, *for some* $A \subseteq N$
2. $\theta_1(x) \geq \theta_2(x) \geq \cdots \geq \theta_{2^n-1}(x)$

Definition

The nucleolus solution is the solution $\nu : \mathcal{G}(N) \to \mathbb{R}^n$ such that $\nu(v)$ is the set of the imputations x such that $\theta(x) \leq_L \theta(y)$, for all y imputations of the game v

Remark

$x \leq_L y$ if either $x = y$ or there exists $j \geq 1$ such that $x_i = y_i$ for all $i < j$, and $x_j < y_j$. $\leq_L$ defines a total order in any Euclidean space

An example

Example

Three players, $v(A) = 1$ if $|A| \geq 2$, 0 otherwise.
Suppose $x = (a, b, 1 - a - b)$, with $a, b \geq 0$ and $a + b \leq 1$. The coalitions S complaining ($e(S, \emptyset) > 0$) are those with two members.

$$e(\{1,2\}) = 1 - (a + b), e(\{1,3\}) = b, e(\{2,3\}) = a$$

We must minimize

$$\max\{1 - a - b, b, a\}$$

$\nu = (1/3, 1/3, 1/3)$

Remember $C(v) = \emptyset$

Nucleolus: one point solution

Theorem

For every TU game v with nonmepty imputation set, the nucleolus $\nu(v)$ is a singleton

Thus the nucleolus is a one point solution

Nucleolus in the core

Proposition

Suppose v is such that $C(v) \neq \emptyset$. *Then* $\nu(v) \in C(v)$

Proof Take $x \in C(v)$. Then $\theta_1(x) \leq 0$. Thus $\theta_1(\nu)(v) \leq 0$. Then $\nu(v) \in C(v)$ ■

Another example

$$v(\{1\}) = a, v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0,\ v(\{1,2\}) = b, v(\{1,3\}) = c,\ v(N) = c$$

$$C(v) = \{(x, 0, c - x) :\ b \leq x \leq c\}$$

Must find x: $\nu(v) = (x, 0, c - x)$. The relevant excesses are

$$e(\{1,2\}) = b - x, \quad e(\{2,3\}) = x - c$$

Thus

$$\nu(v) = \{\frac{b+c}{2}, 0, \frac{c-b}{2}\}$$