The Shapley value and power indices

Roberto Lucchetti

Politecnico di Milano
Summary of the slides

1. The Shapley value
2. The axioms and the theorem
3. The Shapley value in simple games
4. Semivalues
5. The UN security council
Properties for a one point solution

Let $\phi : \mathcal{G}(N) \rightarrow \mathbb{R}^n$ be a one point solution

A list of properties to be fulfilled by a one point solution

1. For every $v \in \mathcal{G}(N)$ \( \sum_{i \in N} \phi_i(v) = v(N) \)
2. Let $v \in \mathcal{G}(N)$ be a game with the following property, for players $i, j$: for every $A$ not containing $i, j$, $v(A \cup \{i\}) = v(A \cup \{j\})$. Then $\phi_i(v) = \phi_j(v)$
3. Let $v \in \mathcal{G}(N)$ and $i \in N$ be such that $v(A) = v(A \cup \{i\})$ for all $A$. Then $\phi_i(v) = 0$
4. for every $v, w \in \mathcal{G}(N)$, $\phi(v + w) = \phi(v) + \phi(w)$
Property 1) is **efficiency**

Property 2) is **symmetry**: symmetric players must take the same

Property 3) is **Null player property**: a player contributing nothing to any coalition must have nothing

Property 4) is **additivity**
Consider the following function $\sigma : \mathcal{G}(N) \rightarrow \mathbb{R}^n$

$$\sigma_i(v) = \sum_{S \in 2^N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} \left[ v(S \cup \{i\}) - v(S) \right]$$

Then $\sigma$ is the only function $\phi$ fulfilling properties of efficiency, symmetry, null player and additivity.
The term

\[ m_i(v, S) := v(S \cup \{i\}) - v(S) \]

is called the marginal contribution of player \( i \) to coalition \( S \cup \{i\} \).

The Shapley value is a weighted sum of all marginal contributions of the players.

Interpretation of the weights

Suppose the players plan to meet in a certain room at a fixed hour, and suppose the expected arrival time is the same for all players. If player \( i \) enters into the coalition \( S \) if and only at her arrival in the room she finds all members of \( S \) and only them, the probability to join coalition \( S \) is

\[
\frac{s!(n - s - 1)!}{n!}
\]
Proof First step: $\sigma$ fulfills the properties

- Efficiency: $\sum_{i=1}^{n} \sigma_{i}(v) = v(N)$ Consider the generic term $v(S \cup \{i\}) - v(S)$. The term $v(N)$ appears, only with positive coefficient $n$ times, once for every player, when $S = N \setminus \{i\}$. Its coefficient is $\frac{(n-1)!(n-n)!}{n!} = \frac{1}{n}$.

Now, let $A \neq N$; the term $v(A)$ appears both with positive and negative coefficients:

- the positive coefficient $\frac{(a-1)!(n-a)!}{n!}$ appears $a$ times, one for every player $i \in A$ setting in the formula $S = A \setminus \{i\}$: its total sum is thus $\frac{a!(n-a)!}{n!}$

- the negative coefficient $-\frac{a!(n-a-1)!}{n!}$ appears $n - a$ times, one for every player $i \notin A$, setting in the formula $S = A$: its total contribution is thus $-\frac{a!(n-a)!}{n!}$

Thus in the sum

$$\sum_{i=1}^{n} \sum_{S \subseteq 2^{N} \setminus \{i\}} \frac{s!(n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)]$$

$v(N)$ appears with coefficient 1 and every $A \neq N$ appears with null coefficient.
Symmetry: if \( v \) is such that there are \( i, j \) such that for every \( A \) not containing \( i, j \), \( v(A \cup \{i\}) = v(A \cup \{j\}) \), then \( \sigma_i(v) = \sigma_j(v) \)

Write

\[
\sigma_i(v) = \sum_{S \in \mathcal{P}(N) \setminus \{i\cup j\}} \frac{s!(n-s-1)!}{n!} \left[ v(S \cup \{i\}) - v(S) \right] + \\
+ \sum_{S \in \mathcal{P}(N) \setminus \{i\cup j\}} \frac{(s+1)!(n-s-2)!}{n!} \left[ v(S \cup \{i \cup j\}) - v(S \cup \{j\}) \right],
\]

\[
\sigma_j(v) = \sum_{S \in \mathcal{P}(N) \setminus \{i\cup j\}} \frac{s!(n-s-1)!}{n!} \left[ v(S \cup \{j\}) - v(S) \right] + \\
+ \sum_{S \in \mathcal{P}(N) \setminus \{i\cup j\}} \frac{(s+1)!(n-s-2)!}{n!} \left[ v(S \cup \{i \cup j\}) - v(S \cup \{i\}) \right]
\]

The terms in the sums are equal for symmetric players

- The null player property is obvious
- The additivity property is obvious
Second step: Uniqueness

1. Given a unanimity game $u_A$:
   - Players not belonging to $A$ are null players: thus $\phi$ assigns zero to them
   - Players in $A$ are symmetric, so $\phi$ assigns the same to them $\phi$ must assign the same amount to both.
   - $\phi$ is efficient

2. From 1 $\phi$ is uniquely determined on the basis of $G(N)$ of the unanimity games

3. The same argument applies to the game $cu_A$, for $c \in \mathbb{R}$

Thus the additivity axiom implies that at most one function fulfills the properties
In the case of the simple games, the Shapley value becomes

\[ \sigma_i(v) = \sum_{A \in \mathcal{A}_i} \frac{a!(n-a-1)!}{n!}, \]

where \( \mathcal{A}_i \) is the set of the coalitions \( A \) such that
- \( i \not\in A \)
- \( A \) is not winning
- \( A \cup \{i\} \) is winning

Alternatively, it can be written:

\[ \sigma_i(v) = \sum_{A \in \mathcal{W}_i} \frac{(a-1)!(n-a)!}{n!}, \]

where \( \mathcal{W}_i \) is the set of the coalitions \( A \) such that
- \( i \in A \)
- \( A \) is winning
- \( A \setminus \{i\} \) is winning
An example

Example

The game:

\[ v(\{1\}) = 0, \ v(\{2\}) = v(\{3\}) = 1, \ v(\{1, 2\}) = 4, \ v(\{1, 3\}) = 4, \ v(\{2, 3\}) = 2, \ v(N) = 8 \]

\[
\begin{array}{ccc}
1 & 2 & 3 \\
123 & 0 & 4 \\
132 & 0 & 4 \\
213 & 3 & 1 \\
231 & 6 & 1 \\
312 & 3 & 4 \\
321 & 6 & 1 \\
\hline
18 & 15 & 15
\end{array}
\]

\[
\sigma_1(v) = \frac{1!1!}{3!} [v(\{1, 2\}) - v(\{2\})] + \frac{1}{6} [v(\{1, 3\}) - v(\{3\})] + \frac{1}{3} [v(N) - v(\{2, 3\})] = 3
\]

\[
\sigma_2(v) = \frac{2}{6} + \frac{5}{6} + \frac{4}{3} = \frac{15}{6}
\]

\[
\sigma_3(v) = \frac{2}{6} + \frac{5}{6} + \frac{4}{3} = \frac{15}{6}
\]

Remark

It was enough to evaluate \( \sigma_1 \) (for instance) the get \( \sigma \)
A simple airport game

Example

The game:

\[ v(\{1\}) = c_1, \ v(\{2\}) = c_2, \ v(\{3\}) = c_3, \ v(\{1, 2\}) = c_2, \ v(\{1, 3\}) = c_3, \ v(\{2, 3\}) = c_3, \ v(N) = c_3 \]

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<td>(\frac{c_1}{3})</td>
<td>(\frac{c_1}{3} + \frac{c_2 - c_1}{2})</td>
<td>(c_3 - \frac{c_2}{2} - \frac{c_1}{6})</td>
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Remark

The first player uses only one km. He equally shares the cost \(c_1\) with the other players. The second km has marginal cost of \(c_2 - c_1\), equally shared by the players 2 and 3 using it, the rest is paid by player 3, the only one using the third km.
In simple games the Shapley value assumes also the meaning of measuring the fraction of power of every player. To measure the relative power of the players in a simple game, the efficiency requirement is not mandatory, and the way coalitions could form can be different from the case of the Shapley value.

**Definition**

A *probabilistic power index* $\psi$ on the set of simple games is

$$
\psi_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} p_i(S) m_i(v, S)
$$

where $p_i$ is a probability measure on $2^{N \setminus \{i\}}$

**Remark**

*Remember: $m_i(v, S) = v(S \cup \{i\}) - v(S)$*
Semivalues

**Definition**

A probabilistic power index $\psi$ on the set of simple games is a *semivalue* if there exists a vector $(p_0, \ldots, p_{n-1})$ such that

$$
\psi_i(v) = \sum_{S \in 2^N \setminus \{i\}} p_s m_i(v, S)
$$

**Remark**

*Since the index is probabilistic, the two conditions must hold*

- $p_s \geq 0$
- $\sum_{n=0}^{n-1} \binom{n-1}{s} p_s = 1$

*If $p_s > 0$ for all $s$, the semivalue is called *regular***
Examples

These are examples of semivalues

- the Shapley value
- the Banzhaf value

\[ \beta_i(v) = \sum_{S \in 2^N \setminus \{i\}} \frac{1}{2^{n-1}} (v(S \cup \{i\}) - v(S)). \]

- the binomial values: \( p_s = q^s (1 - q)^{n-s-1} \), for every \( 0 < q < 1 \)
- the marginal value, \( p_s = 0 \) for \( s = 0, \ldots, n-2 \): \( p_{n-1} = 1 \)
- the dictatorial value \( p_s = 0 \) for \( s = 1, \ldots, n-1 \): \( p_0 = 1 \)
The Shapley value and power indices

The U.N. security council

Example

Let \( N = \{1, \ldots, 15\} \). The permanent members 1, \ldots, 5 are veto players. A resolution passes provided it gets at least 9 votes, including the five votes of the permanent members.

- Let \( i \) be a player which is no veto. His marginal value is 1 if and only if it enters a coalition \( A \) such that \( a = 8 \) and \( A \) contains the 5 veto players. Then

  \[
  \sigma_i = \frac{8! \cdot 6! \cdot 9 \cdot 8 \cdot 7}{15! \cdot 3 \cdot 2} \approx 0.0018648
  \]

- The power of the veto player \( j \) can be calculated by difference and symmetry. The result is \( \sigma_j \approx 0.1962704 \).

Calculating Banzhaf power index

- Let \( i \) be a player which is no veto. Then

  \[
  \beta_i = \frac{1}{2^{14}} \binom{9}{3} = \frac{21}{2^{14}} \approx 0.005127
  \]

- Let \( j \) be a veto player. Then

  \[
  \beta_j = \frac{1}{2^{14}} \left( \binom{10}{4} + \ldots + \binom{10}{10} \right) = \frac{1}{2^{14}} \left( 2^{10} - \sum_{k=0}^{3} \binom{10}{k} \right) = \frac{53}{2^{10}} \approx 0.0517578
  \]

Remark

- The ratio \( \frac{\sigma_i}{\sigma_j} \approx 105.25 \)
- The ratio \( \frac{\beta_i}{\beta_j} \approx 10.0951 \)