Cooperative games (1)

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Summary of the slides

1. Cooperative game
2. TU game
3. Examples
4. Additive, superadditive and convex games
5. Imputation
6. Core
7. Nucleolus

Cooperative game

Definition

A cooperative game is $(N, V : \mathcal{P}(N) \to \mathbb{R}^n)$ *where*

$$V(A) \subseteq \mathbb{R}^A$$

- $\mathcal{P}(N)$ is the collection of all nonempty subsets of the finite set N, such that $|N| = n$, the set of the players
- $V(A)$, for a given $A \in \mathcal{P}(N)$ is the set of the aggregate utilities of the players in coalition A: $x = (x_i)_{i \in A} \in V(A)$ if the players in A, acting by themselves in the game, can guarantee utility x_i to every $i \in A$
- Sometimes $V(A)$ represents costs rather utilities, in this case of course all inequalities must be reversed

TU game

Definition

A transferable utility game (TU game) is a function

$$v : 2^N \to \mathbb{R}$$

such that $v(\emptyset) = 0$

TU game is a cooperative game:

$$V(A) = \{x \in \mathbb{R}^A : \sum_{i \in A} x_i \leq v(A)\}$$

Seller and Buyers

Example

There are one seller and two potential buyers for an important, indivisible good. Player one, the seller, evaluates the good a. Players two and three evaluate it b and c, respectively. $a < b < c$

The game

$$\begin{cases} v(\{1\}) = a, v(\{2\}) = v(\{3\}) = 0 \\ v(\{1,2\}) = b, v(\{1,3\}) = c, v(\{2,3\}) = 0, \\ v(N) = c \end{cases}$$

What can we expect it will happen?

Glove game

Example

N players are have a glove each, some of them a right glove, some other a left glove. Aim is to have pairs of gloves.

To formalize: a partition $\{L, R\}$ of N is assigned

$$v(S) = \min\{|S \cap L|, |S \cap R|\}$$

How the players will form pairs of gloves?

Case Player 1 and 2 with right glove, Player 3 with left glove:
The game

$$\begin{cases} v(\{1\}) = v(\{2\}) = v(\{3\}) = 0 \\ v(\{1,2\}) = 0, v(\{1,3\}) = v(\{2,3\}) = 1 \\ v(N) = 1 \end{cases}$$

Children game

Example

Three players must vote a name of one of them. If one of the players will get at least two votes, she will get 1000 Euros. They can make binding agreements about how sharing money. In case no one gets two nominations, the 1000 Euros are lost

In this case

$$\begin{cases} v(A) = 1000 & \text{if } |A| \geq 2 \\ v(A) = 0 & \text{otherwise} \end{cases}$$

How the money could be divided among players?

Weighted majority game

Example

The game $[q; w_1, \dots, w_n]$ is to provide a model of the situation where n parties in a Parliament must take a decision. Party i has w_i members; to be approved a proposal needs at least q votes

$$v(A) = \begin{cases} 1 & \sum_{I \in A} w_i \geq q \\ 0 & otherwise \end{cases}$$

In the old UN council a decision needs the favorable vote of the 5 permanent members plus at least 4 of the other 10 non permanent

$$v = [39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

Can we quantify the relative power of each party?

Bankruptcy game

Definition

A bankruptcy game is defined by the triple $B = (N, c, E)$, where $N = \{1, \dots, n\}$ is the set of creditors, $c = \{c_1, \dots, c_n\}$ where c_i represents the credit claimed by player i and E is the estate. The bankruptcy condition is then $E < \sum_{i \in N} c_i = C$

$$v_P(S) = \max\left(0, E - \sum_{i \in N \setminus S} c_i\right) \quad S \subseteq N$$

Less realistic

$$v_O(S) = \min\left(E, \sum_{i \in S} c_i\right) \quad S \subseteq N$$

How is it fair to divide E among claimants?

Airport game

Definition

There is a group N of Flying companies needing a new landing lane in a city. N is partitioned into groups $N_1, N_2, \dots, N_k$ such that to each N_j, is associated the cost $c_j < c_{j+1}$ to build the landing lane

$$v(S) = \max\{c_i : i \in S\}$$

How can we share the total cost c_k among the companies?

Peer games

Let $N = \{1, \dots, n\}$ be the set of players and $T = (N, A)$ a directed rooted tree. Each agent i has an individual potential v_i which represents the gain that player i can generate if all players at an upper level in the hierarchy cooperate with him.

For every $i \in N$, we denote by $S(i)$ the set of all agents in the unique directed path connecting 1 to i, i.e. the set of superiors of i

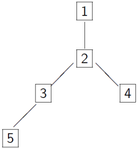
Definition

The peer game is the game v such that

$$v(S) = \sum_{i \in N : S(i) \subseteq S} a_i$$

How should we divide the value $v(N)$ among the players?

A peer game



$$v(A) = 0 \text{ } if \text{ } 1 \notin A, v(A) = v_1 \text{ } if \text{ } 2 \notin A$$

$$v(\{1,2\}) = v(\{1,2,5\}) = v_1 + v_2$$

$$v(\{1,2,4\}) = v(\{1,2,4,5\}) = v_1 + v_2 + v_4$$

$$v(\{1,2,3,4\}) = v_1 + v_2 + v_3 + v_4$$

$$v(\{1,2,3,5\}) = v_1 + v_2 + v_3 + v_5$$

$$v(N) = v_1 + v_2 + v_3 + v_4 + v_5$$

The set of the TU games

Let $\mathcal{G}(N)$ be the set of all cooperative games having N as set of players.
Fix a list $S_1, \dots, S_{2^n-1}$ of coalitions.
A vector $(v_1, \dots, v_{2^n-1})$ represents a game, setting $v_i = v(S_i)$. Thus

Proposition

$\mathcal{G}(N)$ is isomorphic to $\mathbb{R}^{2^n-1}$

Proposition

the set $\{u_A : A \subseteq N\}$ of the unanimity games u_A

$$u_A(T) = \begin{cases} 1 & \text{if} \quad A \subseteq T \\ 0 & \textit{otherwise} \end{cases}$$

is a basis for the space $\mathcal{G}(N)$

Interesting subsets of $\mathcal{G}(N)$

Definition

A game is said to be additive if

$$v(A \cup B) = v(A) + v(B)$$

when $A \cap B = \emptyset$ *, it is said superadditive if*

$$v(A \cup B) \geq v(A) + v(B)$$

when $A \cap B = \emptyset$

The set of the additive games is a vector space of dimension n. All games introduced in the Examples are superadditive. Superadditive games are games where the grand coalition forms

Convex games

Definition

A game is said to be convex if

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$$

The airport game is a convex game, convex games are superadditive.
The children game is not convex, since

$$v(\{1,2\}) + v(\{2,3\}) = 1 + 1 > 1 + 0 = v(N) + v(\{2\})$$

Simple games

Definition

A game $v \in G$ is called simple provided

- $v(S) \in \{0,1\}$ *for every nonempty coalition* S
- $A \subseteq C$ *implies* $v(A) \leq v(C)$
- $v(N) = 1$

$v(A) = 1$ means the coalition A is winning, otherwise it is loosing

Weighted majority games are simple games, unanimity games are simple games. Simple games are characterized by the list of all minimal winning coalitions

Definition

A coalition A in the simple game v is called minimal winning coalition if

- $v(A) = 1$
- $B \subsetneq A$ *implies* $v(B) = 0$

Solutions of cooperative games

Definition

A solution vector for the game $v \in \mathcal{G}(N)$ is a vector $(x_1, \ldots, x_n)$. A solution concept (briefly, solution) for the game $v \in \mathcal{G}(N)$ is a multifunction

$$S : \mathcal{G}(N) \to \mathbb{R}^n$$

The solution vector $x = (x_1, \ldots, x_n)$ assigns utility x_i to player i (cost in case v represents costs). A solution assigns to every game a set (maybe empty) of solution vectors

Imputations

Definition

The imputation *solution is the solution* $I : \mathcal{G}(N) \to \mathbb{R}^n$ *such that* $x \in I(v)$ *if*

1. $x_i \geq v(\{i\})$ *for all* i
2. $\sum_{i=1}^{n} x_i = v(N)$

- If a game fulfills $v(N) \geq \sum_i v(\{i\})$ then the imputation solution is nonempty valued
- Efficiency is a mandatory requirement: it makes a real difference with the non cooperative case

The structure of the imputation set

- The imputation set is nonempty if the game is superadditive (condition only sufficient), it reduces to a singleton if the game is additive
- The imputation set lies in the hyperplane $H = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = v(N)\}$ and it is bounded since $x_i \geq v(\{i\})$ for all i. It is the intersection of half spaces, being defined by linear inequalities, thus it is a convex polytope
- If v is additive then $I(v) = \{(v(1), \ldots, v(n)\}$

The core

Definition

The core *is the solution* $C : \mathcal{G}(N) \to \mathbb{R}^n$ *such that*

$$C(v) = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = v(N) \quad \wedge \quad \sum_{i \in S} x_i \geq v(S) \quad \forall S \subseteq N \right\}$$

Imputations are efficient distributions of utilities accepted by all players individually, core vectors are efficient distributions of utilities accepted by all coalitions

Efficiency is a mandatory requirement: it makes a real difference with the non cooperative case

The core of some games

Seller-Buyers

$$v(\{1\}) = a, v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0,\ v(\{1,2\}) = b, v(\{1,3\}) = c,\ v(N) = c$$

$$\begin{cases} x_1 \geq a, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq b, x_1 + x_3 \geq c, x_1 + x_3 \geq 0 \\ x_1 + x_2 + x_3 = c \end{cases}$$

$$C(v) = \{(x, 0, c - x) : \quad b \leq x \leq c\}$$

The glove game, the particular case

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,2\}) = 0, v(\{1,3\}) = v(\{2,3\}) = v(N) = 1$$

$$\begin{cases} x_1, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq 0, x_1 + x_3 \geq 1, x_2 + x_3 \geq 1 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

$$C(v) = \{(0,0,1)\}$$

This can be extended to any glove game: if l people have left gloves and r people, $r > l$, have right gloves, then

$$C(v) = \{\underbrace{1,\dots,1}_{l \text{ times}}, \underbrace{0,\dots,0}_{r \text{ times}}\}$$

Children game

$v(A) = 1$ if $|A| \geq 2$, zero otherwise

$$\begin{cases} x_1, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq 1, x_1 + x_3 \geq 1, x_2 + x_3 \geq 1 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

$C(v) = \emptyset$

The structure of the core

Proposition

The core $C(v)$ is a polytope (i.e. the smallest closed convex set containing a finite number of points)

Same proof as for the imputation set. The core reduces to the singleton $(v(\{1\}), \dots, v(\{n\}))$ if v is additive; it can be empty also for a superadditive game

The core in simple games

Definition

In a game v, a player i is a veto player *if $v(A) = 0$ for all A such that $i \notin A$*

Theorem

Let v be a simple game. Then $C(v) \neq \emptyset$ if and only if there is at least one veto player. When a veto player exists, the core is the closed convex polytope with extreme points the vectors $(0, \dots, 1, \dots, 0)$ where the 1 corresponds to a veto player

Proof If there is no veto player, then for every i $N \setminus \{i\}$ is a winning coalition. Suppose $(x_1, \dots, x_n) \in C(v)$

$$\sum_{j \neq i} x_j = 1; \quad i = 1, \dots, n$$

Summing up the above inequalities

$$(n-1) \sum_{j=1}^{n} x_j = n$$

a contradiction since $\sum_{j=1}^{n} x_j = 1$. Conversely, any imputation assigning zero to the non-veto players is in the core ■

The core in convex games

Proposition

Let v be a convex game. Then $C(v) \neq \emptyset$

Proof It can be checked that the vector $(x_1, \dots, x_n)$ is in the core of v, where $x_1 = v(\{1\})$ and for $i > 1$ $x_i = v(\{1, \dots, i\}) - v(\{1, \dots, i-1\})$ belongs to the core ■

Nonemptyness of the core: equivalent formulation

Given a game v, consider the following LP problem

$$\min \sum_{i=1}^{n} x_i \tag{1}$$

$$\sum_{i \in S} x_i \geq v(S) \;\; \textit{for all } S \subseteq N \tag{2}$$

Theorem

The above LP problem (1),(2) has always a nonempty set of solution C. The core $C(v)$ is nonempty if and only if the optimal value of the LP is $v(N)$. In case the optimal value of the LP is $v(N)$, then $C(v) = C$

Remark

The value V of the LP is $V \geq v(N)$, due to the constraint in (2) $\sum_i x_i \geq v(N)$; thus for every x fulfilling (2) it is $\sum_{i=1}^n x_i \geq v(N)$

Dual formulation

Theorem

$C(v) \neq \emptyset$ if and only if every vector $(\lambda_S)_{S\subseteq N}$ fulfilling the conditions

$$\lambda_S \geq 0 \ \forall S \subseteq N \quad \textit{and}$$

$$\sum_{S:i\in S\subseteq N} \lambda_S = 1 \quad \textit{for all } i = 1, \dots, n$$

verifies also:

$$\sum_{S\subseteq N} \lambda_S v(S) \leq v(N)$$

Proof

Proof The LP problem (1),(2) associated to the core problem has the following matrix form

$$\begin{cases} \min\langle c, x\rangle : \\ Ax \geq b \end{cases}$$

where c, A, b are the following objects:

$$c = 1_n, \qquad b = (v(\{1\}), v(\{2\}), \dots, v(\{1,2\}), \dots, v(N))$$

and A is a $2^n - 1 \times n$ matrix with the following features

1. it is boolean (i.e. made by 0's and 1's)
2. the 1's in the row j are in correspondence with the players in S_j

The dual of the problem takes the form

$$\begin{cases} \max \sum_{S\subseteq N} \lambda_S v(S) \\ \lambda_S \geq 0 \\ \sum_{S:i\in S\subseteq N} \lambda_S = 1 \qquad \textit{for all } i \end{cases}$$

Since the primal does have solutions, the fundamental duality theorem states that also the dual has solution, and there is no duality gap. Thus the core $C(v)$ is nonempty if and only if the value V of the dual problem is such that $V \leq v(N)$ ∎

Example: three players

LP

$$\min x_1 + x_2 + x_3 :$$

$$x_i \geq v(\{i\}) \; i = 1, 2, 3$$

$$x_1 + x_2 \geq v(\{1, 2\})$$

$$x_1 + x_3 \geq v(\{1, 3\})$$

$$x_2 + x_3 \geq v(\{2, 3\})$$

$$x_1 + x_2 + x_3 \geq v(N)$$

DUAL

$$\max \left[\lambda_{\{1\}} v(\{1\}) + \lambda_{\{2\}} v(\{2\}) + \lambda_{\{3\}} v(\{3\}) + + \lambda_{\{1,2\}} v(\{1,2\}) + \lambda_{\{1,3\}} v(\{1,3\}) + \lambda_{\{2,3\}} v(\{2,3\}) + \lambda_N v(N) \right] :$$

$$\lambda_S \geq 0 \; \forall S$$

$$\lambda_{\{1\}} + \lambda_{\{1,2\}} + \lambda_{\{1,3\}} + \lambda_N = 1$$

$$\lambda_{\{2\}} + \lambda_{\{1,2\}} + \lambda_{\{2,3\}} + \lambda_N = 1$$

$$\lambda_{\{3\}} + \lambda_{\{1,3\}} + \lambda_{\{2,3\}} + \lambda_N = 1$$

Extreme points of the constraint set

Definition

A family $(S_1, \dots, S_m)$ of coalitions is called balanced provided there exists $\lambda = (\lambda_1, \dots, \lambda_m)$ such that $\lambda_i > 0 \; \forall i = 1, \dots, m$ and, for all $i \in N$

$$\sum_{k:i \in S_k} \lambda_k = 1$$

λ is called a balancing vector of the family

Example

- *A partition of N is a balancing family, with balancing vector made by all 1*
- *Let $N = \{1, 2, 3, 4\}$. The family $(\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4\})$ is balanced with balancing vector $(1/2, 1/2, 1/2, 1)$.*
- *$(\{1\}, \{2\}, \{3\}, N)$ is balanced for $N = \{1, 2, 3\}$, and every vector of the form $(p, p, p, 1 - p)$, $0 < p < 1$, is a balancing vector.*
- *The family $(\{1, 2\}, \{1, 3\}, \{3\})$ is not balanced*

Cont'd

Remark

Given a vector $\lambda = (\lambda)_S$ fulfilling the inequalities defining the dual constraint set

$$\begin{cases} \lambda_S \geq 0 \\ \sum_{S: i \in S \subseteq N} \lambda_S = 1 \quad \textit{for all } i \end{cases}$$

the positive coefficients in it are the balancing vectors of a balanced family

Let $N = \{1, 2, 3\}$

- $\{\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}\}$ corresponds to the balanced family $\mathcal{B} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{N\}$
- $\{0, 0, 0, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}\}$ corresponds to the balanced family $\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{N\}\}$

Cont'd

Definition

A minimal balancing family is a balancing family such that no subfamily of the family is balanced

Lemma

A balanced family is minimal if and only if its balancing vector is unique

Theorem

The positive coefficient of the extreme points of the constraint set

$$\begin{cases} \lambda_S \geq 0 \\ \sum_{S:i\in S\subseteq N} \lambda_S = 1 \quad \textit{for all } i \end{cases}$$

are the balancing vectors of the minimal balanced coalitions

Conclusion

To find the extreme points of the dual constraint set it is enough to find balancing families with unique balancing vector

Remark

The partitions of N are minimal balanced families. The condition related to them

$$\sum_{S \subseteq N} \lambda_S v(S) \leq v(N)$$

is automatically fulfilled if the game is super additive

The three player case

The minimal balancing families

$$\begin{array}{ccc}
(1,1,1,0,0,0,0) & \textit{with balanced family} & (\{1\},\{2\},\{3\}) \\
(1,0,0,0,0,1,0) & \textit{with balanced family} & (\{1\},\{2,3\}) \\
(0,1,0,0,1,0,0) & \textit{with balanced family} & (\{2\},\{1,3\}) \\
(0,0,1,1,0,0,0) & \textit{with balanced family} & (\{3\},\{1,2\}) \\
(0,0,0,0,0,0,1) & \textit{with balanced family} & (N), \\
(0,0,0,(1/2),(1/2),(1/2),0) & \textit{with balanced family} & (\{1,2\},\{1,3\},\{2,3\})
\end{array}$$

Only the last one corresponds to a balanced family not being a partition
If the game is super additive,only one condition is to be checked: the core is non empty provided

$$v(\{1,2\})+v(\{1,3\})+v(\{2,3\}) \leq 2v(N)$$

Final remarks

- There are superadditive games v such that $C(v) = \emptyset$
- There are non superadditive games v such that $C(v) \neq \emptyset$

Excess

A TU game v is given

Definition

The excess of a coalition A over the imputation x is

$$e(A,x) = v(A) - \sum_{i \in A} x_i$$

$e(A,x)$ is a measure of the dissatisfaction of the coalition A with respect to the assignment of the imputation x

Remark

An imputation x of the game v belongs to $C(v)$ if and only if $e(A,x) \leq 0$ for all A

Definition

The lexicographic vector attached to the imputation x is the vector

$$\theta(x) = (\theta_1(x), \theta_2(x), \ldots, \theta_{2^n-1}(x))$$

such that

1. $\theta_i(x) = e(A, x)$, *for some* $A \subseteq N$
2. $\theta_1(x) \geq \theta_2(x) \geq \cdots \geq \theta_{2^n-1}(x)$

Definition

The nucleolus solution is the solution $\nu : \mathcal{G}(N) \to \mathbb{R}^n$ such that $\nu(v)$ is the set of the imputations x verifying $\theta(x) \leq_L \theta(y)$, for all y imputations of the game v

Remark

$x \leq_L y$ if either $x = y$ or there exists $j \geq 1$ such that $x_i = y_i$ for all $i < j$, and $x_j < y_j$. $\leq_L$ defines a total order in any Euclidean space

An example

Example

Three players, $v(A) = 1$ *if* $|A| \geq 2$*, 0 otherwise.*
Suppose $x = (a, b, 1 - a - b)$*, with* $a, b \geq 0$ *and* $a + b \leq 1$*. The coalitions* S *complaining* $(e(S, \emptyset) > 0)$ *are those with two members.*

$$e(\{1,2\}) = 1 - (a + b), e(\{1,3\}) = b, e(\{2,3\}) = a$$

We must minimize

$$\max\{1 - a - b, b, a\}$$

$\nu = (1/3, 1/3, 1/3)$

Remember $C(v) = \emptyset$

Nucleolus: one point solution

Theorem

For every TU game v with nonmepty imputation set, the nucleolus $\nu(v)$ is a singleton

The proof of the above proposition is not immediate, since it can happen that $\theta(x) = \theta(y)$ for imputations $x \neq y$. The proof shows that this cannot happen if they are minimal lexicographically

Thus the nucleolus is a one point solution

Nucleolus in the core

Proposition

Suppose v is such that $C(v) \neq \emptyset$. Then $\nu(v) \in C(v)$

Proof Take $x \in C(v)$. Then $\theta_1(x) \leq 0$. Thus $\theta_1(\nu)(v) \leq 0$. Then $\nu(v) \in C(v)$ ■

Another example

$$v(\{1\}) = a, v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0,\ v(\{1,2\}) = b, v(\{1,3\}) = c,\ v(N) = c$$

$$C(v) = \{(x, 0, c - x) :\ b \leq x \leq c\}$$

Must find x: $\nu(v) = (x, 0, c - x)$. The relevant excesses are

$$e(\{1,2\}) = b - x, \quad e(\{2,3\}) = x - c$$

Thus

$$\nu(v) = \{\frac{b+c}{2}, 0, \frac{c-b}{2}\}$$