Cooperative games (1)

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Summary of the slides

1. Cooperative game
2. TU game
3. Examples
4. Additive, superadditive and convex games
5. Imputation
6. Core

Cooperative game

Definition

A *cooperative game* *is* $(N, V : \mathcal{P}(N) \to \mathbb{R}^n)$ *where*

$$V(A) \subseteq \mathbb{R}^A$$

- $\mathcal{P}(N)$ is the collection of all nonempty subsets of the finite set N, such that $|N| = n$, the set of the players
- $V(A)$, for a given $A \in \mathcal{P}(N)$ is the set of the aggregate utilities of the players in coalition A: $x = (x_i)_{i \in A} \in V(A)$ if the players in A, acting by themselves in the game, can guarantee utility x_i to every $i \in A$
- Sometimes $V(A)$ represents costs rather utilities, in this case of course all inequalities must be reversed

TU game

Definition

A *transferable utility game (TU game)* *is a function*

$$v : 2^N \to \mathbb{R}$$

such that $v(\emptyset) = 0$

TU game is a cooperative game:

$$V(A) = \{x \in \mathbb{R}^A : \sum_{i \in A} x_i \leq v(A)\}$$

Seller and Buyers

Example

There are one seller and two potential buyers for an important, indivisible good. Player one, the seller, evaluates the good a. Players two and three evaluate it b and c, respectively. $a < b < c$

The game

$$\begin{cases} v(\{1\}) = a, v(\{2\}) = v(\{3\}) = 0 \\ v(\{1,2\}) = b, v(\{1,3\}) = c, v(\{2,3\}) = 0, \\ v(N) = c \end{cases}$$

What can we expect it will happen?

Glove game

Example

N players are have a glove each, some of them a right glove, some other a left glove. Aim is to have pairs of gloves.

To formalize: a partition $\{L, R\}$ of N is assigned

$$v(S) = \min\{|S \cap L|, |S \cap R|\}$$

How the players will form pairs of gloves?

Case Player 1 and 2 with right glove, Player 3 with left glove:
The game

$$\begin{cases} v(\{1\}) = v(\{2\}) = v(\{3\}) = 0 \\ v(\{1,2\}) = 0, v(\{1,3\}) = v(\{2,3\}) = 1 \\ v(N) = 1 \end{cases}$$

Children game

Example

Three players must vote a name of one of them. If one of the players will get at least two votes, she will get 1000 *Euros. They can make binding agreements about how sharing money. In case no one gets two nominations, the* 1000 *Euros are lost*

In this case

$$\begin{cases} v(A) = 1000 & \text{if } |A| \geq 2 \\ v(A) = 0 & \text{otherwise} \end{cases}$$

How the money could be divided among players?

Weighted majority game

Example

The game $[q; w_1, \dots, w_n]$ is aimed at providing a model of the situation where n parties in a Parliament must take a decision. Party i has w_i members; to be approved a proposal needs at least q votes

$$v(A) = \begin{cases} 1 & \sum_{I \in A} w_i \geq q \\ 0 & otherwise \end{cases}$$

In the old UN council a decision needs the favorable vote of the 5 permanent members plus at least 4 of the other 10 non permanent

$$v = [39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

How can we quantify the relative power of each party?

Bankruptcy game

Definition

A bankruptcy game is defined by the triple $B = (N, c, E)$*, where* $N = \{1, \dots, n\}$ *is the set of creditors,* $c = \{c_1, \dots, c_n\}$ *where* c_i *represents the credit claimed by player* i *and* E *is the estate. The bankruptcy condition is then* $E < \sum_{i \in N} c_i = C$

$$v_P(S) = \max\left(0, E - \sum_{i \in N \setminus S} c_i\right) \quad S \subseteq N$$

Less realistic

$$v_O(S) = \min\left(E, \sum_{i \in S} c_i\right) \quad S \subseteq N$$

How is it fair to divide E among claimants?

Airport game

Definition

There is a group N of Flying companies needing a new landing lane in a city. N is partitioned into groups $N_1, N_2, \dots, N_k$ such that to each N_j, is associated the cost $c_j < c_{j+1}$ to build the landing lane

$$v(S) = \max\{c_i : i \in S\}$$

How can we share the total cost c_k among the companies?

Peer games

Let $N = \{1, \dots, n\}$ be the set of players and $T = (N, A)$ a directed rooted tree. Each agent i has an individual potential v_i which represents the gain that player i can generate if all players at an upper level in the hierarchy cooperate with him.

For every $i \in N$, we denote by $S(i)$ the set of all agents in the unique directed path connecting 1 to i, i.e. the set of superiors of i

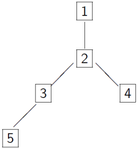
Definition

The peer game is the game v such that

$$v(S) = \sum_{i \in N : S(i) \subseteq S} a_i$$

How should we divide the value $v(N)$ among the players?

A peer game



$$v(A) = 0 \textit{ if } 1 \notin A, v(A) = v_1 \textit{ if } 2 \notin A$$

$$v(\{1,2\}) = v(\{1,2,5\}) = v_1 + v_2$$

$$v(\{1,2,4\}) = v(\{1,2,4,5\}) = v_1 + v_2 + v_4$$

$$v(\{1,2,3,4\}) = v_1 + v_2 + v_3 + v_4$$

$$v(\{1,2,3,5\}) = v_1 + v_2 + v_3 + v_5$$

$$v(N) = v_1 + v_2 + v_3 + v_4 + v_5$$

The set of the TU games

Let $\mathcal{G}(N)$ be the set of all cooperative games having N as set of players.
Fix a list $S_1, \dots, S_{2^n-1}$ of coalitions.
A vector $(v_1, \dots, v_{2^n-1})$ represents a game, setting $v_i = v(S_i)$. Thus

Proposition

$\mathcal{G}(N)$ is isomorphic to $\mathbb{R}^{2^n-1}$

Proposition

the set $\{u_A : A \subseteq N\}$ of the unanimity games u_A

$$u_A(T) = \begin{cases} 1 & \text{if} \quad A \subseteq T \\ 0 & \textit{otherwise} \end{cases}$$

is a basis for the space $\mathcal{G}(N)$

Interesting subsets of $\mathcal{G}(N)$

Definition

A game is said to be additive if

$$v(A \cup B) = v(A) + v(B)$$

when $A \cap B = \emptyset$ *, it is said superadditive if*

$$v(A \cup B) \geq v(A) + v(B)$$

when $A \cap B = \emptyset$

The set of the additive games is a vector space of dimension n. All games introduced in the Examples are superadditive. If a game is superadditive, for the players is convenient to form the grand coalition N.

Simple games

Definition

A game $v \in G$ is called simple provided

- $v(S) \in \{0,1\}$ *for every nonempty coalition* S
- $A \subseteq C$ *implies* $v(A) \leq v(C)$
- $v(N) = 1$

$v(A) = 1$ means the coalition A is winning, otherwise it is losing

Weighted majority games are simple games, unanimity games are simple games. Simple games are characterized by the list of all minimal winning coalitions

Definition

A coalition A in the simple game v is called minimal winning coalition if

- $v(A) = 1$
- $B \subsetneq A$ *implies* $v(B) = 0$

Solutions of cooperative games

Definition

A solution vector for the game $v \in \mathcal{G}(N)$ is a vector $(x_1, \dots, x_n)$. A solution concept (briefly, solution) for the game $v \in \mathcal{G}(N)$ is a multifunction

$$S : \mathcal{G}(N) \to \mathbb{R}^n$$

The solution vector $x = (x_1, \dots, x_n)$ assigns utility x_i to player i (cost in case v represents costs). A solution assigns to every game a set (maybe empty) of solution vectors

Imputations

Definition

The *imputation* *solution is the solution* $I : \mathcal{G}(N) \to \mathbb{R}^n$ *such that* $x \in I(v)$ *if*

1. $x_i \geq v(\{i\})$ *for all* i
2. $\sum_{i=1}^{n} x_i = v(N)$

If a game fulfills $v(N) \geq \sum_i v(\{i\})$ then the imputation solution is nonempty valued

If v is additive then $I(v) = \{(v(1), \ldots, v(n)\}$

The structure of the imputation set

Proposition

The imputation set $I(v)$ is the intersection of half spaces, defined by linear inequalities: it lies in the hyperplane $H = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = v(N)\}$, it is bounded

Boundedness follows from the fact that $x_i \geq v(\{i\})$ for all i and $\sum_{i=1}^n x_i = v(N)$.

- Efficiency is a mandatory requirement: it makes a real difference with the non cooperative case
- The imputation set is nonempty if the game is superadditive (superadditivity of v is only sufficient for non emptyness of $I(v)$)
- The imputation set reduces to a singleton if the game is additive

The core

Definition

The core *is the solution* $C : \mathcal{G}(N) \to \mathbb{R}^n$ *such that*

$$C(v) = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = v(N) \quad \wedge \quad \sum_{i \in S} x_i \geq v(S) \quad \forall S \subseteq N \right\}$$

The core is a subset of the set of imputations. Imputations are efficient distributions of utilities accepted by all players individually, core vectors are efficient distributions of utilities accepted by all coalitions

The core of some games

Seller-Buyers

$$v(\{1\}) = a, v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0,\ v(\{1,2\}) = b, v(\{1,3\}) = c,\ v(N) = c$$

$$\begin{cases} x_1 \geq a, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq b, x_1 + x_3 \geq c, x_1 + x_3 \geq 0 \\ x_1 + x_2 + x_3 = c \end{cases}$$

$$C(v) = \{(x, 0, c - x) : \quad b \leq x \leq c\}$$

The glove game, the particular case

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,2\}) = 0, v(\{1,3\}) = v(\{2,3\}) = v(N) = 1$$

$$\begin{cases} x_1, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq 0, x_1 + x_3 \geq 1, x_2 + x_3 \geq 1 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

$$C(v) = \{(0,0,1)\}$$

This can be extended to any glove game: if l people have left gloves and r people, $r > l$, have right gloves, then

$$C(v) = \{\underbrace{1,\dots,1}_{l \text{ times}}, \underbrace{0,\dots,0}_{r \text{ times}}\}$$

Children game

$v(A) = 1$ if $|A| \geq 2$, zero otherwise

$$\begin{cases} x_1, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq 1, x_1 + x_3 \geq 1, x_2 + x_3 \geq 1 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

$C(v) = \emptyset$

The structure of the core

Proposition

When the core of the game v is nonempty then it is a polytope

Same proof as for the imputation set. The core reduces to the singleton $(v(\{1\}), \dots, v(\{n\}))$ if v is additive; it can be empty also for a superadditive game

The core in simple games

Definition

In a game v, a player i is a veto player if $v(A) = 0$ for all A such that $i \notin A$

Theorem

Let v be a simple game. Then $C(v) \neq \emptyset$ if and only if there is at least one veto player. When a veto player exists, the core is the closed convex polytope with extreme points the vectors $(0, \dots, 1, \dots, 0)$ where the 1 corresponds to a veto player

Proof Suppose i is a non veto player. Then $N \setminus \{i\}$ is a winning coalition. Thus it must be

$$\sum_{j \neq i} x_j \geq 1$$

implying $x_i = 0$ for every $x \in C(v)$. This implies that the core is empty if there is no veto player, and that all distributions of utilities giving 1 to the Veto players is in the core. ■

Nonemptyness of the core: equivalent formulation

Given a game v, consider the following LP problem

$$\min \sum_{i=1}^{n} x_i \tag{1}$$

$$\sum_{i\in S} x_i \geq v(S) \ \ \textit{for all } S \subseteq N \tag{2}$$

Theorem

The above LP problem (1),(2) has always a nonempty set of solution C. The core $C(v)$ is nonempty and $C(v) = C$ if and only if the optimal value of the LP is $v(N)$

Remark

The value V of the LP is $V \geq v(N)$, due to the constraint in (2) $\sum_i x_i \geq v(N)$; thus for every x fulfilling (2) it is $\sum_{i=1}^n x_i \geq v(N)$

Dual formulation

Theorem

$C(v) \neq \emptyset$ if and only if every vector $(\lambda_S)_{S\subseteq N}$ fulfilling the conditions

$$\lambda_S \geq 0 \; \forall S \subseteq N \quad \textit{and}$$

$$\sum_{S: i \in S \subseteq N} \lambda_S = 1 \quad \textit{for all } i = 1, \dots, n$$

verifies also:

$$\sum_{S \subseteq N} \lambda_S v(S) \leq v(N)$$

Proof

Proof The LP problem (1),(2) associated to the core problem has the following matrix form

$$\begin{cases} \min\langle c, x\rangle : \\ Ax \geq b \end{cases}$$

where c, A, b are the following objects:

$$c = 1_n, \qquad b = (v(\{1\}), v(\{2\}), \dots, v(\{1,2\}), \dots, v(N))$$

and A is a $2^n - 1 \times n$ matrix with the following features

1. it is boolean (i.e. made by 0's and 1's)
2. the 1's in the row j are in correspondence with the players in S_j

The dual of the problem takes the form

$$\begin{cases} \max \sum_{S\subseteq N} \lambda_S v(S) \\ \lambda_S \geq 0 \\ \sum_{S: i\in S\subseteq N} \lambda_S = 1 \qquad \textit{for all } i \end{cases}$$

Since the primal does have solutions, the fundamental duality theorem states that also the dual has solution, and there is no duality gap. Thus the core $C(v)$ is nonempty if and only if the value V of the dual problem is such that $V \leq v(N)$ ■

Example: three players

LP

$$\min\ x_1 + x_2 + x_3 :$$

$$x_i \geq v(\{i\})\ i = 1, 2, 3$$

$$x_1 + x_2 \geq v(\{1, 2\})$$

$$x_1 + x_3 \geq v(\{1, 3\})$$

$$x_2 + x_3 \geq v(\{2, 3\})$$

$$x_1 + x_2 + x_3 \geq v(N)$$

DUAL

$$\max \left[\lambda_{\{1\}} v(\{1\}) + \lambda_{\{2\}} v(\{2\}) + \lambda_{\{3\}} v(\{3\}) + +\lambda_{\{1,2\}} v(\{1, 2\}) + \lambda_{\{1,3\}} v(\{1, 3\}) + \lambda_{\{2,3\}} v(\{2, 3\}) + \lambda_N v(N)\right] :$$

$$\lambda_S \geq 0\ \forall S$$

$$\lambda_{\{1\}} + \lambda_{\{1,2\}} + \lambda_{\{1,3\}} + \lambda_N = 1$$

$$\lambda_{\{2\}} + \lambda_{\{1,2\}} + \lambda_{\{2,3\}} + \lambda_N = 1$$

$$\lambda_{\{3\}} + \lambda_{\{1,3\}} + \lambda_{\{2,3\}} + \lambda_N = 1$$

The three player case

The dual formulation of the problem implies that the condition

$$\sum_{S \subseteq N} \lambda_S v(S) \leq v(N)$$

must be verified only on the extreme points of the polytope obtained by imposing the conditions:

$$\begin{cases} \lambda_S \geq 0 & \textit{forallS} \subset N \\ \sum_{S:i \in S \subseteq N} \lambda_S = 1 & \textit{for all } i \end{cases}$$

It can be shown that with three players and if the game is superadditive, only the following condition must be fulfilled

$$v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) \leq 2v(N)$$

Final remarks

- There are superadditive games v such that $C(v) = \emptyset$
- There are non superadditive games v such that $C(v) \neq \emptyset$