The Shapley value and power indices

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Summary of the slides

1. The Shapley value
2. The axioms and the theorem
3. The Shapley value in simple games
4. Semivalues
5. The UN security council

Properties for a one point solution

Let $\phi : \mathcal{G}(N) \to \mathbb{R}^n$ be a one point solution

A list of properties to be fulfilled by a one point solution

1. For every $v \in \mathcal{G}(N)$ $\sum_{i \in N} \phi_i(v) = v(N)$
2. Let $v \in \mathcal{G}(N)$ be a game with the following property, for players i, j: for every A not containing i, j, $v(A \cup \{i\}) = v(A \cup \{j\})$. Then $\phi_i(v) = \phi_j(v)$
3. Let $v \in \mathcal{G}(N)$ and $i \in N$ be such that $v(A) = v(A \cup \{i\})$ for all A. Then $\phi_i(v) = 0$
4. for every $v, w \in \mathcal{G}(N)$, $\phi(v + w) = \phi(v) + \phi(w)$

Comments

1. Property 1) is efficiency
2. Property 2) is symmetry: symmetric players must take the same
3. Property 3) is Null player property: a player contributing nothing to any coalition must have nothing
4. Property 4) is additivity

The Shapley theorem

Theorem

Consider the following function $\sigma : \mathcal{G}(N) \to \mathbb{R}^n$

$$\sigma_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} \frac{s!(n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)]$$

Then σ *is the only function* ϕ *fulfilling properties of efficiency, symmetry, null player and additivity.*

Comments

The term

$$m_i(v,S) := v(S \cup \{i\}) - v(S)$$

is called the marginal contribution of player i to coalition $S \cup \{i\}$

The Shapley value is a weighted sum of all marginal contributions of the players.

Interpretation of the weights

Suppose the players plan to meet in a certain room at a fixed hour, and suppose the expected arrival time is the same for all players. If player i enters into the coalition S if and only at her arrival in the room she finds all members of S and only them, the probability to join coalition S is

$$\frac{s!(n-s-1)!}{n!}$$

Proof(1)

Proof First step: σ fulfills the properties

- Efficiency: $\sum_{i=1}^{n}\sigma_i(v)=v(N)$ Consider the generic term $v(S\cup\{i\})-v(S)$. The term $v(N)$ appears, only with positive coefficient n times, once for every player, when $S=N\setminus\{i\}$. Its coefficient is $\frac{(n-1)!(n-n)!}{n!}=\frac{1}{n}$.

  Now, let $A\neq N$; the term $v(A)$ appears both with positive and negative coefficients:

  - the positive coefficient $\frac{(a-1)!(n-a)!}{n!}$ appears a times, one for every player $i\in A$ setting in the formula $S=A\setminus\{i\}$: its total sum is thus $\frac{a!(n-a)!}{n!}$
  - the negative coefficient -$\frac{a!(n-a-1)!}{n!}$ appears $n-a$ times, one for every player $i\notin A$, setting in the formula $S=A$: its total contribution is thus -$\frac{a!(n-a)!}{n!}$

  Thus in the sum

  $$\sum_{i=1}^{n}\sum_{S\in 2^{N\setminus\{i\}}}\frac{s!(n-s-1)!}{n!}\left[v(S\cup\{i\})-v(S)\right]$$

  $v(N)$ appears with coefficient 1 and every $A\neq N$ appears with null coefficient.

Proof(2)

- Symmetry: if v is such there are i, j such that that for every A not containing i, j, $v(A \cup \{i\}) = v(A \cup \{j\})$, then $\sigma_i(v) = \sigma_j(v)$
  Write
  $$\sigma_i(v) = \sum_{S \in 2^{N \setminus \{i \cup j\}}} \frac{s!(n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)] +$$
  $$+ \sum_{S \in 2^{N \setminus \{i \cup j\}}} \frac{(s+1)!(n-s-2)!}{n!} [v(S \cup \{i \cup j\}) - v(S \cup \{j\})],$$
  $$\sigma_j(v) = \sum_{S \in 2^{N \setminus \{i \cup j\}}} \frac{s!(n-s-1)!}{n!} [v(S \cup \{j\}) - v(S)] +$$
  $$+ \sum_{S \in 2^{N \setminus \{i \cup j\}}} \frac{(s+1)!(n-s-2)!}{n!} [v(S \cup \{i \cup j\}) - v(S \cup \{i\})]$$
  The terms in the sums are equal for symmetric players
- The null player property is obvious
- The additivity property is obvious

Proof(3)

Second step: Uniqueness

Given the unanimity game u_A:

- Players not belonging to A are null players: thus ϕ assigns zero to them
- Players in A are symmetric, so ϕ assigns the same amount to them
- ϕ is efficient

This implies that $\sigma_i(u_A) = 0$ if $i \notin A$, $\sigma_i(u_A) = \frac{1}{a}$ if $i \in A$. More generally, considering the game cu_A, for $c \in \mathbb{R}$ $\sigma_i(u_A) = 0$ if $i \notin A$, $\sigma_i(u_A) = \frac{c}{a}$ if $i \in A$.

Thus ϕ is uniquely determined on the multiple of the elements in one basis of the space of the games. The additivity axiom implies that at most one function fulfills the properties $\blacksquare$

Simple games

In the case of the simple games, the Shapley value becomes

$$\sigma_i(v) = \sum_{S \in \mathcal{S}_i} \frac{s!(n-s-1)!}{n!},$$

where $\mathcal{S}_i$ is the set of the coalitions S such that

- $i \notin S$
- S is a losing coalition
- $S \cup \{i\}$ is a winning coalition

If $S \in \mathcal{S}_i$ we say that i is crucial for S, or also that i is a swing for S

An example

Example

The game:

$$v(\{1\})=0,\ v(\{2\})=v(\{3\})=1,\ v(\{1,2\})=4,\ v(\{1,3\})=4,\ v(\{2,3\})=2,\ v(N)=8$$

	1	*2*	*3*
123	0	4	4
132	0	4	4
213	3	1	4
231	6	1	1
312	3	4	1
321	6	1	1
	$\frac{18}{6}$	$\frac{15}{6}$	$\frac{15}{6}$

$$\sigma_1(v)=\frac{1!1!}{3!}[v(\{1,2\})-v(\{2\})]+\frac{1}{6}[v(\{1,3\})-v(\{3\})]+\frac{1}{3}[v(\{N\})-v(\{2,3\})]=3$$

$$\sigma_2(v)=\frac{2}{6}+\frac{5}{6}+\frac{4}{3}=\frac{15}{6}$$

$$\sigma_3(v)=\frac{2}{6}+\frac{5}{6}+\frac{4}{3}=\frac{15}{6}$$

Remark

It was enough to evaluate σ_1 (for instance) the get σ

A simple airport game

Example

The game:

$$v(\{1\}) = c_1, v(\{2\}) = c_2, v(\{3\}) = c_3, v(\{1,2\}) = c_2, v(\{1,3\}) = c_3, v(\{2,3\}) = c_3, v(N) = c_3$$

	1	*2*	*3*
123	c_1	$c_2 - c_1$	$c_3 - c_2$
132	c_1	0	$c_3 - c_1$
213	0	c_2	$c_3 - c_2$
231	0	c_2	$c_3 - c_2$
312	0	0	c_3
321	0	0	c_3
	$\frac{c_1}{3}$	$\frac{c_1}{3} + \frac{c_2 - c_1}{2}$	$c_3 - \frac{c_2}{2} - \frac{c_1}{6}$

Remark

The first player uses only one km. He equally shares the cost c_1 with the other players. The second km has marginal cost of $c_2 - c_1$, equally shared by the players 2 and 3 using it, the rest is paid by player 3, the only one using the third km

Power indices for simple games

In simple games the Shapley value assumes also the meaning of measuring the fraction of power of every player. To measure the relative power of the players in a simple game, the efficiency requirement is not mandatory, and the way coalitions could form can be different from the case of the Shapley value

Definition

A probabilistic power index ψ on the set of simple games is

$$\psi_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} p_i(S) m_i(v, S)$$

where p_i is a probability measure on $2^{N \setminus \{i\}}$

Remark

Remember: $m_i(v, S) = v(S \cup \{i\}) - v(S)$

Semivalues

Definition

A probabilistic power index ψ on the set of simple games is a semivalue if there exists a vector $(p_0, \dots, p_{n-1})$ such that

$$\psi_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} p_s m_i(v, S)$$

Remark

Since the index is probabilistic, the two conditions must hold

- $p_s \geq 0$
- $\sum_{n=0}^{n-1} \binom{n-1}{s} p_s = 1$

If $p_s > 0$ for all s, the semivalue is called regular

Examples

These are examples of semivalues

- the Shapley value
- the Banzhaf value

$$\beta_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} \frac{1}{2^{n-1}} (v(S \cup \{i\}) - v(S)).$$

- the binomial values: $p_s = q^s (1-q)^{n-s-1}$, for every $0 < q < 1$
- the marginal value, $p_s = 0$ for $s = 0, \ldots, n-2$: $p_{n-1} = 1$
- the dictatorial value $p_s = 0$ for $s = 1, \ldots, n-1$: $p_0 = 1$

The U.N. security council

Example

Let $N = \{1, \dots, 15\}$. The permanent members $1, \dots 5$ are veto players. A resolution passes provided it gets at least 9 votes, including the five votes of the permanent members

- Let i be a player which is no veto. His marginal value is 1 if and only if it enters a coalition A such that $a = 8$ and A contains the 5 veto players. Then
$$\sigma_i = \frac{8! \cdot 6!}{15!}\binom{9}{3} = \frac{8! \cdot 6! \cdot 9 \cdot 8 \cdot 7}{15! \cdot 3 \cdot 2} \simeq 0.0018648$$
- The power of the veto player j can be calculated using efficiency and symmetry. The result is $\sigma_j \simeq 0, 1962704$

Calculating Banzhaf power index

- Let i be a player which is no veto. Then
$$\beta_i = \frac{1}{2^{14}}\binom{9}{3} = \frac{21}{2^{12}} \simeq 0.005127$$
- Let j be a veto player. Then

$$\beta_j = \frac{1}{2^{14}}\left(\binom{10}{4} + \dots \binom{10}{10}\right) = \frac{1}{2^{14}}\left(2^{10} - \sum_{k=0}^{3}\binom{10}{k}\right) = \frac{53}{2^{10}} \simeq 0.0517578$$

Remark

- *The ratio* $\frac{\sigma_i}{\sigma_j} \simeq 105.25$
- *The ratio* $\frac{\beta_i}{\beta_j} \simeq 10.0951$