The nucleolus, the Shapley value and power indices

Roberto Lucchetti

Politecnico di Milano

Excess

A TU game v is given

Definition

The excess of a coalition A over the imputation x is

$$e(A,x) = v(A) - \sum_{i \in A} x_i$$

$e(A,x)$ is a measure of the dissatisfaction of the coalition A with respect to the assignment of the imputation x

Remark

An imputation x of the game v belongs to $C(v)$ if and only if $e(A,x) \leq 0$ for all A

Definition

The lexicographic vector attached to the imputation x is the $(2^n-1)-th$ dimensional vector $\theta(x)$ such that

1. $\theta_i(x) = e(A, x)$, *for some* $A \subseteq N$
2. $\theta_1(x) \geq \theta_2(x) \geq \cdots \geq \theta_{2^n-1}(x)$

Definition

The nucleolus solution is the solution $\nu : \mathcal{G}(N) \to \mathbb{R}^n$ such that $\nu(v)$ is the set of the imputations x such that $\theta(x) \leq_L \theta(y)$, for all y imputations of the game v

Remark

$x \leq_L y$ if either $x = y$ or there exists $j \geq 1$ such that $x_i = y_i$ for all $i < j$, and $x_j < y_j$. $\leq_L$ defines a total order in any Euclidean space

An example

Example

Three players, $v(A) = 1$ if $|A| \geq 2$, 0 otherwise.
Suppose $x = (a, b, 1 - a - b)$, with $a, b \geq 0$ and $a + b \leq 1$. The coalitions S complaining ($e(S, \emptyset) > 0$) are those with two members.

$$e(\{1,2\}) = 1 - (a + b), e(\{1,3\}) = b, e(\{2,3\}) = a$$

We must minimize

$$\max\{1 - a - b, b, a\}$$

$\nu = (1/3, 1/3, 1/3)$

Remember $C(v) = \emptyset$

Nucleolus: one point solution

Theorem

For every TU game v with nonmepty imputation set, the nucleolus $\nu(v)$ is a singleton

Thus the nucleolus is a one point solution

Nucleolus in the core

Proposition

Suppose v is such that $C(v) \neq \emptyset$. Then $\nu(v) \in C(v)$

Proof Take $x \in C(v)$. Then $\theta_1(x) \leq 0$. Thus $\theta_1(\nu)(v) \leq 0$. Then $\nu(v) \in C(v)$ ∎

Another example

$$v(\{1\}) = a, v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0,\ v(\{1,2\}) = b, v(\{1,3\}) = c,\ v(N) = c$$

$$C(v) = \{(x, 0, c - x) :\ b \le x \le c\}$$

Must find x: $\nu(v) = (x, 0, c - x)$. The relevant excesses are

$$e(\{1,2\}) = b - x, \quad e(\{2,3\}) = x - c$$

Thus

$$\nu(v) = \{\frac{b+c}{2}, 0, \frac{c-b}{2}\}$$

Properties for a one point solution

Let $\phi : \mathcal{G}(N) \to \mathbb{R}^n$ be a one point solution

Here is a list of properties ϕ should satisfy

1) For every $v \in \mathcal{G}(N)$

$$\sum_{i \in N} \phi_i(v) = v(N)$$

2) Let $v \in \mathcal{G}(N)$ be a game with the following property, for players i, j: for every A not containing i, j, $v(A \cup \{i\}) = v(A \cup \{j\})$ Then

$$\phi_i(v) = \phi_j(v)$$

3) Let $v \in \mathcal{G}(N)$ and $i \in N$ be such that $v(A) = v(A \cup \{i\})$ for all A. Then

$$\phi_i(v) = 0$$

4) for every $v, w \in \mathcal{G}(N)$, $\phi(v + w) = \phi(v) + \phi(w)$

Comments

- Property 1) is efficiency
- Property 2) is symmetry: symmetric players must take the same
- Property 3) is Null player property: a player contributing nothing to any coalition must have nothing
- Property 4) is additivity

The Shapley theorem

Theorem

Consider the following function $\sigma : \mathcal{G}(N) \to \mathbb{R}^n$

$$\sigma_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} \frac{s!(n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)]$$

Then σ *is the only function* ϕ *fulfilling properties 1),2),3),4)*

Comments

The term

$$m_i(v, S) := v(S \cup \{i\}) - v(S)$$

is called the marginal contribution of player i to coalition $S \cup \{i\}$

The Shapley value is a weighted sum of all marginal contributions of the players.

Interpretation of the weights

Suppose the players plan to meet in a certain room at a fixed hour, and suppose the expected arrival time is the same for all players. If player i enters into the coalition S if and only at her arrival she find in the room all members of S and only them, the probability to join coalition S is

$$\frac{s!(n-s-1)!}{n!}$$

Proof(1)

Proof First step: σ fulfills the properties

- Efficiency: $\sum_{i=1}^{n} \sigma_i(v) = v(N)$

The term $v(N)$ appears n times with coefficient $\frac{(n-1)!(n-n)!}{n!} = \frac{1}{n}$. Let $A \neq N$; the term $v(A)$ appears in the sum a times (once for every player in A), with coefficient

$$\frac{(a-1)!(n-a)!}{n!},$$

providing

$$\frac{a!(n-a)!}{n!}$$

as positive coefficient for $v(A)$. $v(A)$ appears with negative sign $n-a$ times (once for each player not in A) with coefficient

$$\frac{a!(n-a-1)!}{n!}$$

and the result is

$$\frac{a!(n-a)!}{n!}$$

Thus every $A \neq N$ appears with null coefficient in the sum

Proof(2)

- Symmetry. Suppose v is such that for every A not containing i, j, $v(A \cup \{i\}) = v(A \cup \{j\})$. We must then prove $\sigma_i(v) = \sigma_j(v)$. Write

$$\sigma_i(v) = \sum_{S \in 2^{N \setminus \{i \cup j\}}} \frac{s!(n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)] +$$

$$+ \sum_{S \in 2^{N \setminus \{i \cup j\}}} \frac{(s+1)!(n-s-2)!}{n!} [v(S \cup \{i \cup j\}) - v(S \cup \{j\})],$$

$$\sigma_j(v) = \sum_{S \in 2^{N \setminus \{i \cup j\}}} \frac{s!(n-s-1)!}{n!} [v(S \cup \{j\}) - v(S)] +$$

$$+ \sum_{S \in 2^{N \setminus \{i \cup j\}}} \frac{(s+1)!(n-s-2)!}{n!} [v(S \cup \{i \cup j\}) - v(S \cup \{i\})]$$

  The terms in the sums are equal for symmetric players
- The null player property is obvious
- The linearity property is obvious

Proof(3)

Second step: Uniqueness

Consider a unanimity game u_A.

- Players not belonging to A are null players: thus ϕ assigns zero to them
- Players in A are symmetric, so ϕ assigns the same to them ϕ must assign the same amount to both.
- ϕ is efficient

Then ϕ is uniquely determined on the basis of $\mathcal{G}(N)$ of the unanimity games

The same argument applies to the game cu_A, for $c \in \mathbb{R}$

The additivity axiom implies that at most one function fulfills the properties $\blacksquare$

Simple games

In the case of the simple games, the Shapley value becomes

$$\sigma_i(v) = \sum_{A \in \mathcal{A}_i} \frac{(a-1)!(n-a)!}{n!},$$

where $\mathcal{A}_i$ is the set of the coalitions A such that

- $i \in A$
- A is winning
- $A \setminus \{i\}$ is not winning

An example

Example

The game:

$$v(\{1\}) = 0,\ v(\{2\}) = v(\{3\}) = 1,\ v(\{1,2\}) = 4,\ v(\{1,3\}) = 4,\ v(\{2,3\}) = 2,\ v(N) = 8$$

	1	*2*	*3*
123	0	4	4
132	0	4	4
213	3	1	4
231	6	1	1
312	3	4	1
321	6	1	1
	$\frac{18}{6}$	$\frac{15}{6}$	$\frac{15}{6}$

$$\sigma_1(v) = \frac{1!1!}{3!}[v(\{1,2\}) - v(\{2\})] + \frac{1}{6}[v(\{1,3\}) - v(\{3\})] + \frac{1}{3}[v(\{N\}) - v(\{2,3\})] = 3$$

$$\sigma_2(v) = \frac{2}{6} + \frac{5}{6} + \frac{4}{3} = \frac{15}{6}$$

$$\sigma_3(v) = \frac{2}{6} + \frac{5}{6} + \frac{4}{3} = \frac{15}{6}$$

Remark

It was enough to evaluate σ_1 (for instance) the get σ

A simple airport game

Example

The game:

$$v(\{1\}) = 0,\ v(\{2\}) = v(\{3\}) = 1,\ v(\{1,2\}) = 4,\ v(\{1,3\}) = 4,\ v(\{2,3\}) = 2,\ v(N) = 8$$

	1	*2*	*3*
123	c_1	$c_2 - c_1$	$c_3 - c_2$
132	c_1	0	$c_3 - c_1$
213	0	c_2	$c_3 - c_2$
231	0	c_2	$c_3 - c_2$
312	0	0	c_3
321	0	0	c_3
	$\frac{c_1}{3}$	$\frac{c_1}{3} + \frac{c_2 - c_1}{2}$	$c_3 - \frac{c_2}{2} - \frac{c_1}{6}$

Remark

The first player uses only one km. He equally shares the cost c_1 with the other players. The secondo km has a marginal cost of $c_2 - c_1$, equally shared by the players using it, the rest is paid by the player, the only one using the third km

Power indices

In simple games the Shapley value assumes also the meaning of measuring the fraction of power of every player. To measure the relative power of the players in a simple game, the efficiency requirement is not anymore mandatory, and the way coalitions could form can be different from the case of the Shapley value

Definition

A probabilistic power index ψ on the set of simple games is

$$\psi_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} p_i(S) m_i(v, S)$$

where p_i is a probability measure on $2^{N \setminus \{i\}}$

Remark

Remember: $m_i(v, S) = v(S \cup \{i\}) - v(S)$

Semivalues

Definition

A probabilistic power index ψ on the set of simple games is a semivalue if there exists a vector $(p_0, \dots, p_{n-1}$ such that

$$\psi_i(v) = \sum_{n=0}^{n-1} \binom{n-1}{s} p_s m_i(v, S)$$

Remark

Since the index is probabilistic, the two conditions must hold

- $p_s \geq 0$
- $\sum_{n=0}^{n-1} \binom{n-1}{s} p_s = 1$

If $p_s > 0$ for all s, the semivalue is called regular

Examples

These are examples of semivalues

- the Shapley value
- the Banzhaf value

$$\beta_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} \frac{1}{2^{n-1}}(v(S \cup \{i\}) - v(S)).$$

- Binomial values: $p_s = q^s(1-q)^{n-s-1}$, for every $0 < q < 1$
- the marginal value, $p_s = 0$ for $s = 0, \dots, n-2$: $p_{n-1} = 1$
- the dictatorial value $p_s = 0$ for $s = 1, \dots, n-1$: $p_0 = 1$

The U.N. security council

Example

Let $N = \{1, \dots, 15\}$. The permanent members $1, \dots 5$ are veto players. A resolution passes provided it gets at least 9 votes, including the five votes of the permanent members

- Let i be a player which is no veto. His marginal value is 1 if and only if it enters a coalition A such that $a = 8$ and A contains the 5 veto players. Then
$$\sigma_i = \frac{8! \cdot 6! \cdot 9 \cdot 8 \cdot 7}{15! \cdot 3 \cdot 2} \simeq 0.0018648$$
- The power of the veto player j can be calculated by difference and symmetry. The result is $\sigma_j \simeq 0, 1962704$

Calculating Banzhaf power index

- Let i be a player which is no veto. Then
$$\beta_i = \frac{1}{2^{14}}\binom{9}{3} = \frac{21}{2^{12}} \simeq 0.005127$$
- Let j be a veto player. Then

$$\beta_j = \frac{1}{2^{14}}\left(\binom{10}{4} + \dots \binom{10}{10}\right) = \frac{1}{2^{14}}\left(2^{10} - \sum_{k=0}^{3}\binom{10}{k}\right) = \frac{53}{2^{10}} \simeq 0.0517578$$

Remark

- *The ratio* $\frac{\sigma_i}{\sigma_j} \simeq 105.25$
- *The ratio* $\frac{\beta_i}{\beta_j} \simeq 10.0951$