

# Cooperative games (1)

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# Cooperative game

## Definition

A *cooperative game* is  $(N, V : \mathcal{P}(N) \rightarrow \mathbb{R}^n)$  where

$$V(A) \subseteq \mathbb{R}^A$$

$\mathcal{P}(N)$  is the collection of all nonempty subsets of the finite set  $N$ , such that  $|N| = n$ , the set of the players

$V(A)$ , for a given  $A \in \mathcal{P}(N)$  is the set of the aggregate utilities of the players in coalition  $A$

$x = (x_i)_{i \in A} \in V(A)$  if the players in  $A$ , acting by themselves in the game, can guarantee utility  $x_i$  to every  $i \in A$

# TU game

## Definition

A *transferable utility game (TU game)* is a function

$$v : 2^N \rightarrow \mathbb{R}$$

such that  $v(\emptyset) = 0$

TU game is a cooperative game

$$V(A) = \{x \in \mathbb{R}^A : \sum_{i \in A} x_i \leq v(A)\}$$

# Seller and Buyers

## Example

*There are one seller and two potential buyers for an important, indivisible good. Player one, the seller, evaluates the good  $a$ . Players two and three evaluate it  $b$  and  $c$ , respectively.  $a < b < c$*

The game

$$\begin{cases} v(\{1\}) = a, v(\{2\}) = v(\{3\}) = 0 \\ v(\{1, 2\}) = b, v(\{1, 3\}) = c, v(\{2, 3\}) = 0, \\ v(N) = c \end{cases}$$

What can we expect it will happen?

# Glove game

## Example

*$N$  players are have a glove each, some of them a right glove, some other a left glove. Aim is to have pairs of gloves.*

To formalize: a partition  $\{L, R\}$  of  $N$  is assigned

$$v(S) = \min\{|S \cap L|, |S \cap R|\}$$

How the players will form pairs of gloves?

Case Player 1 and 2 with right glove, Player 3 with left glove:

The game

$$\begin{cases} v(\{1\}) = v(\{2\}) = v(\{3\}) = 0 \\ v(\{1, 2\}) = 0, v(\{1, 3\}) = v(\{2, 3\}) = 1 \\ v(N) = 1 \end{cases}$$

# Children game

## Example

*Three players must vote a name of one of them. If one of the players will get at least two votes, she will get 1000 Euros. They can make binding agreements about how sharing money. If no one gets two nominations, the 1000 Euros are lost*

In this case  $v(A) = 1000$  if  $|A| \geq 2$ , zero otherwise

How the money could be divided among players?

# Weighted majority game

## Example

*The game  $[q; w_1, \dots, w_n]$  models the situation where  $n$  parties in a Parliament must take a decision. Party  $i$  has  $w_i$  members, and at least  $q$  votes are needed to pass a resolution*

$$v(A) = \begin{cases} 1 & \sum_{i \in A} w_i \geq q \\ 0 & \text{otherwise} \end{cases}$$

In the old UN council the votes of 5 permanent members plus at least 4 of the other 10 non permanent members are needed for a resolution to pass

$$v = [39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

Can we quantify the relative power of each party?

# Bankruptcy game

## Definition

A bankruptcy game is defined by the triple  $B = (N, c, E)$ , where  $N = \{1, \dots, n\}$  is the set of creditors,  $c = \{c_1, \dots, c_n\}$  is the vector of the credits claimed by them and  $E$  is the available capital. The bankruptcy condition is then  $E < \sum_{i \in N} c_i = C$

$$v_P(S) = \max \left( 0, E - \sum_{i \in N \setminus S} c_i \right) \quad S \subseteq N$$

Less realistic

$$v_O(S) = \min \left( E, \sum_{i \in S} c_i \right) \quad S \subseteq N$$

How is it fair to divide  $E$  among claimants?



# Airport game

## Definition

*There is a group  $N$  of Flying companies needing a new landing lane in a city.  $N$  is partitioned into groups  $N_1, N_2, \dots, N_k$  such that to each  $N_j$  is associated the cost  $c_j \leq c_{j+1}$  to build the landing lane*

$$v(S) = \max\{c_i : i \in S\}$$

How can we share the total cost  $c_k$  among the companies?

# Peer games

Let  $N = \{1, \dots, n\}$  be the set of players and  $T = (N, A)$  a directed rooted tree. Each agent  $i$  has an individual potential  $v_i$  which represents the gain that player  $i$  can generate if all players at an upper level in the hierarchy cooperate with him.

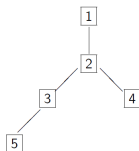
For every  $i \in N$ , we denote by  $S(i)$  the set of all agents in the unique directed path connecting 1 to  $i$ , i.e. the set of superiors of  $i$

## Definition

The *peer game* is the game  $v$  such that

$$v(S) = \sum_{i \in N: S(i) \subseteq S} v_i$$

## Peer games



$$v(A) = 0 \text{ if } 1 \notin A, v(A) = v_1 \text{ if } 2 \notin A$$

$$v(\{1, 2\}) = v(\{1, 2, 5\}) = v_1 + v_2$$

$$v(\{1, 2, 4\}) = v(\{1, 2, 4, 5\}) = v_1 + v_2 + v_4$$

$$v(\{1, 2, 3, 4\}) = v_1 + v_2 + v_3 + v_4$$

$$v(\{1, 2, 3, 5\}) = v_1 + v_2 + v_3 + v_5$$

$$v(N) = v_1 + v_2 + v_3 + v_4 + v_5$$

# The set of the TU games

## Proposition

Let  $\mathcal{G}(N)$  be the set of all cooperative games having  $N$  as set of players.  $\mathcal{G}(N)$  is isomorphic to  $\mathbb{R}^{2^n - 1}$  and the set  $\{u_T : T \subseteq N\}$  of the **unanimity games**  $u_T$

$$u_T(A) = \begin{cases} 1 & \text{if } T \subseteq A \\ 0 & \text{otherwise} \end{cases}$$

is a basis for the space  $\mathcal{G}(N)$

**Proof** Writing

$$v = \sum_{T \subseteq N} c_T u_T$$

the constants  $c_T$  are inductively defined:  $c_{\{i\}} = v(\{i\})$  and, for  $T \subseteq N$ ,  $t \geq 2$

$$c_T = v(T) - \sum_{S \subset T} c_S$$

For every coalition  $A$  it holds

$$\sum_{T \subseteq N} c_T u_T(A) = \sum_{T \subseteq A} c_T = c_A + \sum_{S \subset A} c_S = v(A) - \sum_{S \subset A} c_S + \sum_{S \subset A} c_S = v(A) \quad \blacksquare$$

# Interesting subsets of $\mathcal{G}(N)$

## Definition

A game is said to be **additive** if

$$v(A \cup B) = v(A) + v(B)$$

when  $A \cap B = \emptyset$  **superadditive** if

$$v(A \cup B) \geq v(A) + v(B)$$

when  $A \cap B = \emptyset$

The set of the additive games is a **vector space of dimension  $n$** . All games introduced in the Examples are superadditive. If a game is superadditive, then all players have interest in forming the grand coalition  $N$ .

# Convex games

## Definition

A game is said to be *convex* if

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$$

The airport game and the bankruptcy games are convex games; convex games are superadditive. The children game is not convex

# Simple games

## Definition

A game  $v \in G$  is called **simple** provided  $v$  is valued on  $\{0, 1\}$ ,  $A \subseteq C$  implies  $v(A) \leq v(C)$  and  $v(N) = 1$

$v(A) = 1$  means the coalition  $A$  is **winning**, otherwise it is **losing**

Weighted majority games are simple games, unanimity games are simple games. Simple games are characterized by the list of all minimal winning coalitions

## Definition

A coalition  $A$  in the simple game  $v$  is called **minimal winning coalition** if

- $v(A) = 1$
- $B \subsetneq A$  implies  $v(B) = 0$

# Solutions of cooperative games

## Definition

A **solution vector** for the game  $v \in \mathcal{G}(N)$  is a vector  $(x_1, \dots, x_n)$ . A **solution concept** (briefly, **solution**) for the game  $v \in \mathcal{G}(N)$  is a multifunction

$$S : \mathcal{G}(N) \rightarrow \mathbb{R}^n$$

The solution vector  $x = (x_1, \dots, x_n)$  assigns utility  $x_i$  to player  $i$  (cost in case  $v$  represents costs). A solution assigns to every game a set (maybe empty) of solution vectors



# Imputations

## Definition

The *imputation* solution is the solution  $I : \mathcal{G}(N) \rightarrow \mathbb{R}^n$  such that  $x \in I(v)$  if

- 1  $x_i \geq v(\{i\})$  for all  $i$
- 2  $\sum_{i=1}^n x_i = v(N)$

Efficiency is a mandatory requirement: it makes a real difference with the non cooperative case

If a game fulfills  $v(N) \geq \sum_i v(\{i\})$  then the imputation solution is nonempty valued.

# The structure of the imputation set

## The imputation set

- lies in the hyperplane  $H = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = v(N)\}$
- it is bounded since  $x_i \geq v(\{i\})$  for all  $i$
- it is the intersection of half spaces inside  $H$ , since it is defined in terms of linear inequalities, thus it is the smallest convex set containing a finite number of points, i.e. a polytope
- it is nonempty if the game is superadditive, it reduces to a singleton if the game is additive

# The core

## Definition

The *core* is the solution  $C : \mathcal{G}(N) \rightarrow \mathbb{R}^n$  such that

$$C(v) = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i = v(N) \quad \wedge \quad \sum_{i \in S} x_i \geq v(S) \quad \forall S \subseteq N \right\}$$

Imputations are efficient distributions of utilities accepted by players **individually**, core vectors are **efficient** distributions of utilities accepted by **all coalitions**

# The core of some games

## Seller-Buyers

$$v(\{1\}) = a, v(\{2\}) = v(\{3\}) = v(\{2, 3\}) = 0, v(\{1, 2\}) = b, v(\{1, 3\}) = c, v(N) = c$$

$$\begin{cases} x_1 \geq a, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq b, x_1 + x_3 \geq c, x_1 + x_3 \geq 0 \\ x_1 + x_2 + x_3 = c \end{cases}$$

$$C(v) = \{(x, 0, c - x) : b \leq x \leq c\}$$

# A glove game with three players

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1, 2\}) = 0, v(\{1, 3\}) = v(\{2, 3\}) = v(N) = 1$$

$$\begin{cases} x_1, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq 0, x_1 + x_3 \geq 1, x_2 + x_3 \geq 1 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

$$C(v) = \{(0, 0, 1)\}$$

This can be extended to any glove game

## Children game

$v(A) = 1$  if  $|A| \geq 2$ , zero otherwise

$$\begin{cases} x_1, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq 1, x_1 + x_3 \geq 1, x_2 + x_3 \geq 1 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

$$C(v) = \emptyset$$

# The structure of the core

## Proposition

*The core  $C(v)$ , whenever is nonempty, is a polytope*

Same proof as for the imputation set.

- the core is the **singleton**  $(v(\{1\}), \dots, v(\{n\}))$  if  $v$  is additive
- it can be empty also for a superadditive game

# The core in simple games

## Definition

In a game  $v$ , a player  $i$  is a **veto player** if  $v(A) = 0$  for all  $A$  such that  $i \notin A$

## Theorem

Let  $v$  be a simple game. Then  $C(v) = \emptyset$  if there is no veto player. If there is at least a veto player the core is the convex polytope with extreme points the vectors  $(0, \dots, 1, \dots, 0)$  where the 1 corresponds to a VP

**Proof** If there is no veto player, then for every  $i$  there is  $N \setminus \{i\}$  is a winning coalition. Suppose  $(x_1, \dots, x_n) \in C(v)$ , then  $\sum_{j \neq i} x_j = 1$  for all  $i = 1, \dots, n$ . Summing up the above inequalities

$$\sum_{i=1}^n \sum_{j \neq i} x_j = (n-1) \sum_{j=1}^n x_j = n$$

a contradiction since  $\sum_{j=1}^n x_j = 1$ . Conversely, any imputation assigning zero to all the non-veto players is in the core ■



# The core in convex games

## Proposition

*Let  $v$  be a convex game. Then  $C(v) \neq \emptyset$*

**Proof** It can be checked that the vector  $(x_1, \dots, x_n)$  is in the core of  $v$ , where

- $x_1 = v(\{1\})$
- $x_i = v(\{1, \dots, i\}) - v(\{1, \dots, i-1\})$  for  $i > 1$

Actually the core is the convex hull of the vectors built as above considering all permutations of the players. ■

# Nonemptiness of the core: equivalent formulation

Given a game  $v$ , consider the following LP problem

$$\min \sum_{i=1}^n x_i \quad (1)$$

$$\sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N \quad (2)$$

## Theorem

*The above LP problem (1),(2) has always a nonempty set of solution  $C$ . The core  $C(v)$  is nonempty and  $C(v) = C$  if and only if the value of the LP is  $v(N)$*

## Remark

*The value  $V$  of the LP is  $V \geq v(N)$ , due to the constraint in (2)*  
 $\sum_i x_i \geq v(N)$

# Dual formulation

## Theorem

$C(v) \neq \emptyset$  if and only if every vector  $(\lambda_S)_{S \subseteq N}$  fulfilling the conditions

$$\lambda_S \geq 0 \quad \forall S \subseteq N \quad \text{and}$$

$$\sum_{S: i \in S \subseteq N} \lambda_S = 1 \quad \text{for all } i = 1, \dots, n$$

verifies also:

$$\sum_{S \subseteq N} \lambda_S v(S) \leq v(N)$$

## Proof

**Proof** The LP problem (1),(2) associated to the core problem has the following matrix form

$$\begin{cases} \min \langle c, x \rangle : \\ Ax \geq b \end{cases}$$

where  $c, A, b$  are the following objects:

$$c = \mathbf{1}_n, \quad b = (v(\{1\}), v(\{2\}), v(\{3\}), \dots, v(N))$$

and  $A$  is a  $2^n - 1 \times n$  matrix with the following features

- 1 it is boolean (i.e. made by 0's and 1's)
- 2 the 1's in the row  $j$  are in correspondence with the players in  $S_j$

The dual of the problem takes the form

$$\begin{cases} \max \sum_{S \subseteq N} \lambda_S v(S) \\ \lambda_S \geq 0 \\ \sum_{S: i \in S} \lambda_S = 1 \quad \text{for all } i \end{cases}$$

Since the primal does have solutions, the fundamental duality theorem states that also the dual has solution, and there is no duality gap. Thus the core  $C(v)$  is nonempty if and only if the value  $V$  of the dual problem is such that  $V \leq v(N)$  ■

# Example: three players

LP

$$\begin{aligned}
 & \min x_1 + x_2 + x_3 : \\
 & x_i \geq v(\{i\}) \quad i = 1, 2, 3 \\
 & x_1 + x_2 \geq v(\{1, 2\}) \\
 & x_1 + x_3 \geq v(\{1, 3\}) \\
 & x_2 + x_3 \geq v(\{2, 3\}) \\
 & x_1 + x_2 + x_3 \geq v(N)
 \end{aligned}$$

DUAL

$$\begin{aligned}
 & \max [\lambda_{\{1\}} v(\{1\}) + \lambda_{\{2\}} v(\{2\}) + \lambda_{\{3\}} v(\{3\}) + \lambda_{\{1,2\}} v(\{1, 2\}) + \lambda_{\{1,3\}} v(\{1, 3\}) + \lambda_{\{2,3\}} v(\{2, 3\}) + \lambda_N v(N)] : \\
 & \lambda_S \geq 0 \quad \forall S \\
 & \lambda_{\{1\}} + \lambda_{\{1,2\}} + \lambda_{\{1,3\}} + \lambda_N = 1 \\
 & \lambda_{\{2\}} + \lambda_{\{1,2\}} + \lambda_{\{2,3\}} + \lambda_N = 1 \\
 & \lambda_{\{3\}} + \lambda_{\{1,3\}} + \lambda_{\{2,3\}} + \lambda_N = 1
 \end{aligned}$$

# Extreme points of the constraint set

## Definition

A family  $(S_1, \dots, S_m)$  of coalitions is called **balanced** provided there exists  $\lambda = (\lambda_1, \dots, \lambda_m)$  such that  $\lambda_i > 0 \forall i = 1, \dots, m$  and, for all  $i \in N$

$$\sum_{k:i \in S_k} \lambda_k = 1$$

$\lambda$  is called a **balancing vector** of the family

## Example

A **partition** of  $N$  is a balancing family, with balancing vector made by all 1. The family  $(\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4\})$  is balanced with balancing vector  $(1/2, 1/2, 1/2, 1)$ .  $(\{1\}, \{2\}, \{3\}, N)$  is balanced and every vector of the form  $(p, p, p, 1 - p)$ ,  $0 < p < 1$ , is a balancing vector. The family  $(\{1, 2\}, \{1, 3\}, \{3\})$  is not balanced

## Cont'd

## Remark

Given a vector  $\lambda = (\lambda_S)$  fulfilling the inequalities defining the dual constraint set

$$\begin{cases} \lambda_S \geq 0 \\ \sum_{S:i \in S \subseteq N} \lambda_S = 1 \quad \text{for all } i \end{cases}$$

the positive coefficients in it are the balancing vectors of a balanced family

Let  $N = \{1, 2, 3\}$

- $\{\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}\}$  corresponds to the balanced family  $\mathcal{B} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{N\}\}$
- $\{0, 0, 0, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}\}$  corresponds to the balanced family  $\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{N\}\}$

## Cont'd

## Definition

A *minimal* balancing family is a balancing family such that no subfamily of the family is balanced

## Lemma

A balanced family is minimal if and only if its balancing vector is unique

## Theorem

The positive coefficients of the extreme points of the constraint set

$$\begin{cases} \lambda_S \geq 0 \\ \sum_{S:i \in S \subseteq N} \lambda_S = 1 \quad \text{for all } i \end{cases}$$

are the balancing vectors of the minimal balanced coalitions



# Conclusion

To find the extreme points of the dual constraint set it is enough to find **balancing families with unique balancing vector**

## Remark

*The partitions of  $N$  are minimal balanced families. The condition related to them*

$$\sum_{S \subseteq N} \lambda_S v(S) \leq v(N)$$

*is automatically fulfilled if the game is super additive*

# The three player case

The minimal balancing families

$(1, 1, 1, 0, 0, 0, 0)$  with balanced family  $(\{1\}, \{2\}, \{3\})$

$(1, 0, 0, 0, 0, 1, 0)$  with balanced family  $(\{1\}, \{2, 3\})$

$(0, 1, 0, 0, 1, 0, 0)$  with balanced family  $(\{2\}, \{1, 3\})$

$(0, 0, 1, 1, 0, 0, 0)$  with balanced family  $(\{3\}, \{1, 2\})$

$(0, 0, 0, 0, 0, 0, 1)$  with balanced family  $(N)$

$(0, 0, 0, (1/2), (1/2), (1/2), 0)$  with balanced family  $(\{1, 2\}, \{1, 3\}, \{2, 3\})$

Only the last one corresponds to a balanced family not being a partition. If the game is super additive, only one condition is to be checked: the core is non empty provided

$$v(\{1, 2\}) + v(\{1, 3\}) + v(\{2, 3\}) \leq 2v(N)$$

The number of extreme points grows rapidly w.r.t. the number of players. F.i. for superadditive games of 4 players thirteen conditions must be checked.