Cooperative games (1)

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Cooperative game

Definition

A cooperative game is $(N, V : \mathcal{P}(N) \to \mathbb{R}^n)$ *where*

$$v(A) \subseteq \mathbb{R}^A$$

$\mathcal{P}(N)$ is the collection of all nonempty subsets of the finite set N, such that $|N| = n$, the set of the players

$V(A)$, for a given $A \in \mathcal{P}(N)$ is the set of the aggregate utilities of the players in coalition A

$x = (x_i)_{i \in A} \in V(A)$ if the players in A, acting by themselves in the game, can guarantee utility x_i to every $i \in A$

TU game

Definition

A transferable utility game (TU game) is a function

$$v : 2^N \to \mathbb{R}$$

such that $v(\emptyset) = 0$

TU game is a cooperative game

$$V(A) = \{x \in \mathbb{R}^A : \sum_{i \in A} x_i \leq v(A)\}$$

Buyers and sellers games

Example

(Buyers and sellers)

1) *There are one seller and two potential buyers for an important, indivisible good. Player one, the seller, evaluates the good a. Players two and three evaluate it b and c, respectively. $a < b < c$*
2) *There are two sellers and one potential buyer for an important, indivisible good*

1) The game

$$v(\{1\}) = a, v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0, \ v(\{1,2\}) = b, v(\{1,3\}) = c, \ v(N) = c$$

2) The game

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,2\}) = 0, v(\{1,3\}) = v(\{2,3\}) = v(N) = 1$$

What can we expect we will happen in these situations?

Children game

Example

Three players must vote a name of one of them. If one of the players will get at least two votes, she will get 1000 *Euros. They can make binding agreements about how sharing money. if no one gets two nominations, the* 1000 *Euros are lost*

In this case $v(A) = 1000$ if $|A| \geq 2$, zero otherwise

How the money could be divided among players?

Weighted majority game

Example

The game $[q; w_1, \ldots, w_n]$ is to provide a model of the situation where n parties in a Parliament must take a decision. Party i has w_i members, and a law to be approved needs at least q votes

$$v(A) = \begin{cases} 1 & \sum_{I \in A} w_i \geq q \\ 0 & otherwise \end{cases}$$

In the old UN council a decision need, to pass, the vote of the 5 permnent members plus at least 4 of the other 10 non permanent

$$v = [39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

Can we quantify the relative power of each party?

Bankruptcy game

Definition

A bankruptcy game is defined by the triple $B = (N, c, E)$, where $N = \{1, \dots, n\}$ is the set of creditors, $c = \{c_1, \dots, c_n\}$ is the credits claimed by them and E is the available capital. The bankruptcy condition is then $E < \sum_{i \in N} c_i = C$

$$v_P(S) = \max\left(0, E - \sum_{i \in N \setminus S} c_i\right) \quad S \subseteq N$$

Less realistic

$$v_O(S) = \min\left(E, \sum_{i \in S} c_i\right) \quad S \subseteq N$$

How is it fair to divide E among claimants?

Glove game

Example

N players are have a glove each, some of them a right glove, some other a left glove. Aim is to have pairs of gloves.

A partition $\{L, R\}$ of N is assigned

$$v(S) = \min\{|S \cap L|, |S \cap R|\}$$

How the players will form pairs of gloves?

Airport game

Definition

There is a group N of Flying companies needing a new landing lane in a city. N is partitioned into groups $N_1, N_2, \ldots, N_k$ such that to each N_j, is associated the cost $c_j \leq c_{j+1}$ to build the landing lane

$v(S) = \max\{c_i : i \in S\}$

How can we share the total cost c_k among the companies?

Peer games

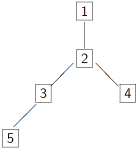
Let $N = \{1, \dots, n\}$ be the set of players and $T = (N, A)$ a directed rooted tree. Each agent i has an individual potential v_i which represents the gain that player i can generate if all players at an upper level in the hierarchy cooperate with him.

For every $i \in N$, we denote by $S(i)$ the set of all agents in the unique directed path connecting 1 to i, i.e. the set of superiors of i

Definition

The peer game is the game v such that

$$v(S) = \sum_{i \in N : S(i) \subseteq S} a_i$$



$$v(A) = 0 \textit{ if } 1 \notin A, v(A) = v_1 \textit{ if } 2 \notin A$$

$$v(\{1,2\}) = v(\{1,2,5\}) = v_1 + v_2$$

$$v(\{1,2,4\}) = v(\{1,2,4,5\}) = v_1 + v_2 + v_4$$

$$v(\{1,2,3,4\}) = v_1 + v_2 + v_3 + v_4$$

$$v(\{1,2,3,5\}) = v_1 + v_2 + v_3 + v_5$$

$$v(N) = v_1 + v_2 + v_3 + v_4 + v_5$$

The set of the TU games

Let $\mathcal{G}(N)$ be the set of all cooperative games having N as set of players. Fix a list $S_1, \dots, S_{2^n-1}$ of coalitions. A vector $(v_1, \dots, v_{2^n-1})$ represents a game, setting $v_i = v(S_i)$. Thus

Proposition

$\mathcal{G}(N)$ is isomorphic to $\mathbb{R}^{2^n-1}$

Proposition

the set $\{u_A : A \subseteq N\}$ of the unanimity games u_A

$$u_A(T) = \begin{cases} 1 & \text{if} \quad A \subseteq T \\ 0 & \textit{otherwise} \end{cases}$$

is a basis for the space $\mathcal{G}(N)$

Interesting subsets of $\mathcal{G}(N)$

Definition

A game is said to be additive if

$$v(A \cup B) = v(A) + v(B)$$

when $A \cap B = \emptyset$ *superadditive if*

$$v(A \cup B) \geq v(A) + v(B)$$

when $A \cap B = \emptyset$

The set of the additive games is a vector space of dimension n. All games introduced in the Examples are superadditive. Superadditive games are games where the grand coalition forms

Convex games

Definition

A game is said to be convex if

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$$

The airport game is a convex game, convex games are superadditive.
The children game is not superadditive

Simple games

Definition

A game $v \in G$ is called simple provided v is valued on $\{0,1\}$, $A \subseteq C$ implies $v(A) \leq v(C)$ and $v(N) = 1$

$v(A) = 1$ means the coalition A is winning, otherwise it is loosing

Weighted majority games are simple games, unanimity games are simple games. Simple games are characterized by the list of all minimal winning coalitions

Definition

A coalition A in the simple game v is called minimal winning coalition if

- $v(A) = 1$
- $B \subsetneq A$ *implies* $v(B) = 0$

Solutions of cooperative games

Definition

A solution vector for the game $v \in \mathcal{G}(N)$ is a vector $(x_1, \dots, x_n)$. A solution concept (briefly, solution) for the game $v \in \mathcal{G}(N)$ is a multifunction

$$S : \mathcal{G}(N) \to \mathbb{R}^n$$

The solution vector $x = (x_1, \dots, x_n)$ assigns utility x_i to player i (cost in case v represents costs). A solution assigns to every game a set (maybe empty) of solution vectors

Imputations

Definition

The *imputation* *solution is the solution* $I : \mathcal{G}(N) \to \mathbb{R}^n$ *such that* $x \in I(v)$ *if*

1. $x_i \geq v(\{i\})$ *for all* i
2. $\sum_{i=1}^{n} x_i = v(N)$

If a game fulfills $v(N) \geq \sum_i v(\{i\})$ then the imputation solution is nonempty valued

If v is additive then $I(v) = \{(v(1), \ldots, v(n))\}$

The structure of the imputation set

Proposition

The imputation set $I(v)$ is a polytope (i.e. the smallest closed convex set containing a finite number of points)

The imputation set lies in the hyperplane $H = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = v(N)\}$ and it is bounded since $x_i \geq v(\{i\})$ for all i. It is the intersection of half spaces, being defined by linear inequalities. The imputation set is nonempty if the game is superadditive, it reduces to a singleton if the game is additive

The core

Definition

The core is the solution $C : \mathcal{G}(N) \to \mathbb{R}^n$ such that

$$C(v) = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = v(N) \quad \wedge \quad \sum_{i \in S} x_i \geq v(S) \quad \forall S \subseteq N \right\}$$

Imputations are efficient distributions of utilities accepted by all players individually, core vectors are efficient distributions of utilities accepted by all coalitions

Efficiency is a mandatory requirement: it makes a real difference with the non cooperative case

The core of some games

Seller-Buyers

$$v(\{1\}) = a, v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0,\ v(\{1,2\}) = b, v(\{1,3\}) = c,\ v(N) = c$$

$$\begin{cases} x_1 \geq a, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq b, x_1 + x_3 \geq c, x_1 + x_3 \geq 0 \\ x_1 + x_2 + x_3 = c \end{cases}$$

$$C(v) = \{(x, 0, c - x) : \quad b \leq x \leq c\}$$

Sellers-Buyer

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,2\}) = 0, v(\{1,3\}) = v(\{2,3\}) = v(N) = 1$$

$$\begin{cases} x_1, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq 0, x_1 + x_3 \geq 1, x_2 + x_3 \geq 1 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

$$C(v) = \{(0,0,1)\}$$

This can be extended to any glove game

Children game

$v(A) = 1$ if $|A| \geq 2$, zero otherwise

$$\begin{cases} x_1, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq 1, x_1 + x_3 \geq 1, x_2 + x_3 \geq 1 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

$C(v) = \emptyset$

The structure of the core

Proposition

The core $C(v)$ is a polytope (i.e. the smallest closed convex set containing a finite number of points)

Same proof as for the imputation set. The core reduces to the singleton $(v(\{1\}),\dots,v(\{n\}))$ if v is additive; it can be empty also for a superadditive game

The core in simple games

Definition

In a game v, a player i is a veto player if $v(A) = 0$ for all A such that $i \notin A$

Theorem

Let v be a simple game. Then $C(v) \neq \emptyset$ if and only if there is at least one veto player

Proof If there is no veto player, then for every i there is $N \setminus \{i\}$ is a winning coalition. Suppose $(x_1, \dots, x_n) \in C(v)$

$$\sum_{j \neq i} x_j = 1$$

for all $i = 1, \dots, n$. Summing up the above inequalities

$$(n-1) \sum_{j=1}^{n} x_j = n$$

a contradiction since $\sum_{j=1}^{n} x_j = 1$. Conversely, any imputation assigning zero to the non-veto players is in the core ■

The core is the closed convex polytope with extreme points the vectors $(0, \dots, 1, \dots, 0)$ where the 1 corresponds to a VP

The core in convex games

Proposition

Let v be a convex game. Then $C(v) \neq \emptyset$

Proof It can be checked that the vector $(x_1, \dots, x_n)$ is in the core of v, where $x_1 = v(\{1\})$ and for $i > 1$ $x_i = v(\{1, \dots, i\}) - v(\{1, \dots, i-1\})$ belongs to the core ■

Nonemptyness of the core: equivalent formulation

Given a game v, consider the following LP problem

$$\min \sum_{i=1}^{n} x_i \tag{1}$$

$$\sum_{i \in S} x_i \geq v(S) \;\; \textit{for all } S \subseteq N \tag{2}$$

Theorem

The above LP problem (1),(2) has always a nonempty set of solution C. The core $C(v)$ is nonempty and $C(v) = C$ if and only if the value of the LP is $v(N)$

Remark

The value V of the LP is $V \geq v(N)$, due to the constraint in (2) $\sum_i x_i \geq v(N)$; thus for every x fulfilling (2) it is $\sum_{i=1}^{n} x_i \geq v(N)$

Dual formulation

Theorem

$C(v) \neq \emptyset$ if and only if every vector $(\lambda_S)_{S \subseteq N}$ fulfilling the conditions

$$\lambda_S \geq 0 \; \forall S \subseteq N \quad \textit{and}$$

$$\sum_{S: i \in S \subseteq N} \lambda_S = 1 \quad \textit{for all } i = 1, \dots, n$$

verifies also:

$$\sum_{S \subseteq N} \lambda_S v(S) \leq v(N)$$

Proof

Proof The LP problem (1),(2) associated to the core problem has the following matrix form

$$\begin{cases} \min\langle c, x\rangle : \\ Ax \geq b \end{cases}$$

where c, A, b are the following objects:

$$c = 1_n, \qquad b = (v(\{1\}), v(\{2\}), v(\{3\}), \dots, v(N))$$

and A is a $2^n - 1 \times n$ matrix with the following features

1. it is boolean (i.e. made by 0's and 1's)
2. the 1's in the row j are in correspondence with the players in S_j

The dual of the problem takes the form

$$\begin{cases} \max \sum_{S\subseteq N} \lambda_S v(S) \\ \lambda_S \geq 0 \\ \sum_{S:i\in S\subseteq N} \lambda_S = 1 \qquad \textit{for all } i \end{cases}$$

Since the primal does have solutions, the fundamental duality theorem states that also the dual has solution, and there is no duality gap. Thus the core $C(v)$ is nonempty if and only if the value V of the dual problem is such that $V \leq v(N)$ ■

Example: three players

LP

$$\min\ x_1 + x_2 + x_3 :$$

$$x_i \geq v(\{i\})\ i = 1,2,3$$

$$x_1 + x_2 \geq v(\{1,2\})$$

$$x_1 + x_3 \geq v(\{1,3\})$$

$$x_2 + x_3 \geq v(\{2,3\})$$

$$x_1 + x_2 + x_3 \geq v(N)$$

DUAL

$$\max\left[\lambda_{\{1\}} v(\{1\}) + \lambda_{\{2\}} v(\{2\}) + \lambda_{\{3\}} v(\{3\}) + +\lambda_{\{1,2\}} v(\{1,2\}) + \lambda_{\{1,3\}} v(\{1,3\}) + \lambda_{\{2,3\}} v(\{2,3\}) + \lambda_N v(N)\right] :$$

$$\lambda_S \geq 0\ \forall S$$

$$\lambda_{\{1\}} + \lambda_{\{1,2\}} + \lambda_{\{1,3\}} + \lambda_N = 1$$

$$\lambda_{\{2\}} + \lambda_{\{1,2\}} + \lambda_{\{2,3\}} + \lambda_N = 1$$

$$\lambda_{\{3\}} + \lambda_{\{1,3\}} + \lambda_{\{2,3\}} + \lambda_N = 1$$

Extreme points of the constraint set

Definition

A family $(S_1, \ldots, S_m)$ of coalitions is called balanced provided there exists $\lambda = (\lambda_1, \ldots, \lambda_m)$ such that $\lambda_i > 0 \; \forall i = 1, \ldots, m$ and, for all $i \in N$

$$\sum_{k:i \in S_k} \lambda_k = 1$$

λ is called a balancing vector of the family

Example

A partition of N is a balancing family, with balancing vector made by all 1. The family $(\{1,2\},\{1,3\},\{2,3\},\{4\})$ is balanced for Let $N = \{1,2,3,4\}$, with vector $(1/2, 1/2, 1/2, 1)$. $(\{1\},\{2\},\{3\},N)$ is balanced for $N = \{1,2,3\}$, and every vector of the form $(p,p,p,1-p)$, $0 < p < 1$, is a balancing vector. The family $(\{1,2\},\{1,3\},\{3\})$ is not balanced

Cont'd

Remark

Given a vector $\lambda = (\lambda)_S$ fulfilling the inequalities defining the dual constraint set

$$\begin{cases} \lambda_S \geq 0 \\ \sum_{S: i \in S \subseteq N} \lambda_S = 1 \quad \textit{for all } i \end{cases}$$

the positive coefficients in it are the balancing vectors of a balanced family

Let $N = \{1, 2, 3\}$

- $\{\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{3}{5}\}$ corresponds to the balanced family $\mathcal{B} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{N\}\}$
- $\{0, 0, 0, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{3}{5}\}$ corresponds to the balanced family $\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{N\}\}$

Cont'd

Definition

A minimal balancing family is a balancing family such that no subfamily of the family is balanced

Lemma

A balanced family is minimal if and only if its balancing vector is unique

Theorem

The positive coefficient of the extreme points of the constraint set

$$\begin{cases} \lambda_S \geq 0 \\ \sum_{S:i\in S\subseteq N} \lambda_S = 1 \quad \textit{for all } i \end{cases}$$

are the balancing vectors of the minimal balanced coalitions

Conclusion

To find the extreme points of the dual constraint set it is enough to find balancing families with unique balancing vector

Remark

The partitions of N are minimal balanced families. The condition related to them

$$\sum_{S \subseteq N} \lambda_S v(S) \leq v(N)$$

is automatically fulfilled if the game is super additive

The three player case

The minimal balancing families

$$(1,1,1,0,0,0,0) \quad \textit{with balanced family} \quad (\{1\},\{2\},\{3\})$$
$$(1,0,0,0,0,1,0) \quad \textit{with balanced family} \quad (\{1\},\{2,3\})$$
$$(0,1,0,0,1,0,0) \quad \textit{with balanced family} \quad (\{2\},\{1,3\})$$
$$(0,0,1,1,0,0,0) \quad \textit{with balanced family} \quad (\{3\},\{1,2\})$$
$$(0,0,0,0,0,0,1) \quad \textit{with balanced family} \quad (N),$$
$$(0,0,0,(1/2),(1/2),(1/2),0) \quad \textit{with balanced family} \quad (\{1,2\},\{1,3\},\{2,3\})$$

Only the last one corresponds to a balanced family not being a partition
If the game is super additive,only one condition is to be checked: the core is non empty provided

$$v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) \leq 2v(N)$$