The Nash model

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Summary of the slides

1. Game in strategic form
2. Nash equilibrium profile
3. Existence of Nash equilibrium profiles
4. Existence in mixed strategies for finite game
5. Best reply multifunction
6. Finding mixed equilibria for general finite games
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Definition of non cooperative game

Definition

A two player noncooperative game in strategic form is $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$

X, Y are the strategy sets of the players, f, g their utility functions.

Equilibrium

A Nash equilibrium profile for $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$ is a pair $(\bar{x}, \bar{y}) \in X \times Y$ such that:

- $f(\bar{x}, \bar{y}) \geq f(x, \bar{y})$ for all $x \in X$
- $g(\bar{x}, \bar{y}) \geq g(\bar{x}, y)$ for all $y \in Y$

A Nash equilibrium profile is a joint combination of strategies, stable w.r.t. unilateral deviations of a single player

More than two players

The main ideas of the Nash model can be seen with two players: having more players does not add complexity to the concepts: just notation is more complicated.
Consider an n-player game with strategy sets X_i and payoffs $u_i : X \to \mathbb{R}$ with $X = \prod_{j=1}^{n} X_j$.

Notation: if $x = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$ is a strategy profile, denote by x_{-i} the vector $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ and write also $x = (x_i, x_{-i})$.

Then $\bar{x} = (\bar{x}_i)_{i=1}^{n}$ is a NE p. for the game if for every i, for every $x \in X_i$

$$u_i(\bar{x}) \geq u_i(x, \bar{x}_{-i}).$$

The new rationality paradigm

Observe: new definition of rationality

How does this notion relate to older rationality definitions?

- Dominant strategies
- Backward induction
- Optimal strategies in zero sum games

Dominant strategies
Suppose $\bar{x}$ is a (weakly) dominant strategy for Pl1:

$f(\bar{x}, y) \geq f(x, y)$ for all x, y.

If $\bar{y}$ maximizes the function $y \rightarrowtail g(\bar{x}, y)$

then $(\bar{x}, \bar{y})$ is a NEp

Weakly dominant vs strictly dominant

Suppose $\bar{y}$ maximizes the function $y \rightarrowtail g(\bar{x}, y)$:

- If $\bar{x}$ is a weakly dominant strategy for Pl1, other Nash Equilibria beyond $(\bar{x}, \bar{y})$ can exist
- If $\bar{x}$ is a strictly dominant strategy for PL1, then no other Nash Equilibria exist different from the above one(s)

Nash equilibria in games with perfect information

Backward induction

- Backward induction provides a Nash equilibrium profile for a game of perfect information, since players systematically make an optimal choice in every part of the tree of the game
- It is possible that in games of perfect information there are more equilibria than that one(s) provided by backward induction

Example

Player 1 must claim for himself $x \in [0, 1]$. *Player 2 can either accept* $(1 - x)$ *or decline. If she declines both players get 0, otherwise utilities are* $(x, 1 - x)$

Backward induction provides outcome is $(1, 0)$: strategies

- Propose $x = 1$ for the first player (i.e. he offers nothing to the second player)
- Accept any offer for the second player

On the contrary, any outcome $(x, 1 - x)$ is the result of a NE profile

Nash equilibria in zero sum games

Zero sum games

Theorem

Let X, Y be (nonempty) sets and $f : X \times Y \to \mathbb{R}$ a function. Then the following are equivalent:

1. *The pair $(\bar{x}, \bar{y})$ fulfills*

$$f(x, \bar{y}) \leq f(\bar{x}, \bar{y}) \leq f(\bar{x}, y) \quad \forall x \in X, \ \forall y \in Y$$

2. *The following conditions are satisfied:*
   (i) $\inf_y \sup_x f(x, y) = \sup_x \inf_y f(x, y)$
   (ii) $\inf_y f(\bar{x}, y) = \sup_x \inf_y f(x, y)$
   (iii) $\sup_x f(x, \bar{y}) = \inf_y \sup_x f(x, y)$

Proof

Proof 1) implies 2). From 1)

$$v_2 = \inf_y \sup_x f(x,y) \leq \sup_x f(x,\bar{y}) = f(\bar{x},\bar{y}) = \inf_y f(\bar{x},y) \leq \sup_x \inf_y f(x,y) = v_1$$

Since $v_1 \leq v_2$ always holds, all above inequalities are equalities

Conversely, suppose 2) holds Then

$$\inf_y \sup_x f(x,y) \stackrel{(iii)}{=} \sup_x f(x,\bar{y}) \geq f(\bar{x},\bar{y}) \geq \inf_y f(\bar{x},y) \stackrel{(ii)}{=} \sup_x \inf_y f(x,y)$$

Because of (i), all inequalities are equalities and the proof is complete ■

As a consequence of the theorem

Given a general zero sum game $(X, Y, f : X \times Y \to \mathbb{R})$:

- Any $(\bar{x}, \bar{y})$ Nash equilibrium provides optimal strategies for the players; moreover $f(\bar{x}, \bar{y}) = v$ is the value of the game
- Any pair of optimal strategies: $\bar{x}$ for the first player, $\bar{y}$ for the second player, are such that $(\bar{x}, \bar{y})$ is a NE profile of the game and $f(\bar{x}, \bar{y}) = v$

Existence of Nash equilibria

Denote by BR_1, BR_2 the following multifunctions:

$$BR_1 : Y \to X : \quad BR_1(y) = \mathrm{Arg\ Max}\, \{f(\cdot, y)\}$$

$$BR_2 : X \to Y : \quad BR_2(x) = \mathrm{Arg\ Max}\, \{g(x, \cdot)\}$$

and

$$BR : X \times Y \to X \times Y : \quad BR(x, y) = (BR_1(y), BR_2(x)).$$

$(\bar{x}, \bar{y})$ is a Nash equilibrium for the game if and only if

$$(\bar{x}, \bar{y}) \in BR(\bar{x}, \bar{y})$$

Thus an existence theorem for a Nash equilibrium can be proved using a fixed point theorem.

Kakutani's theorem

Theorem

Let Z be a compact convex subset of an Euclidean space, let $F : Z \to Z$ be such that $F(z)$ is a nonempty closed convex set for all z. Suppose also F has closed graph. Then F has a fixed point: there is $\bar{z} \in Z$ such that $\bar{z} \in F(\bar{z})$

Closed graph means: if $y_n \in F(z_n)$ for all n, if $y_n \to y$ and if $z_n \to z$, then $y \in F(z)$

The Nash theorem

Theorem

Given the game $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$*, suppose:*

- X *and* Y *are compact convex subsets of some Euclidean space*
- f, g *continuous*
- $x \mapsto f(x, y)$ *is quasi concave for all* $y \in Y$
- $y \mapsto g(x, y)$ *is quasi concave for all* $x \in X$

Then the game has an equilibrium

Quasi concavity for a real valued function h means that the sets

$$h_a = \{z : h(z) \geq a\}$$

are convex for all a (maybe empty for some a)

Proof

$BR_1(y)$ and $BR_2(x)$ are nonempty (compactness assumption) closed (continuity of f and g), and convex valued, (quasi concavity)
BR has closed graph: suppose $(u_n, v_n) \in BR(x_n, y_n)$ for all n and $(u_n, v_n) \to (u, v)$, $(x_n, y_n) \to (x, y)$. To prove: $(u, v) \in BR(x, y)$.

We have

$$f(u_n, y_n) \geq f(z, y_n), \qquad g(x_n, v_n) \geq g(x_n, t),$$

for all $z \in X$, $t \in Y$. Taking limits

$$f(u, y) \geq f(z, y), \qquad g(x, v) \geq g(x, t)$$

for every $z \in X$, $t \in Y$ ■

Finite games: notation

Suppose the sets of the strategies of the players are finite, $\{1, \dots, n\}$ for the first player, $\{1, \dots, m\}$ for the second player. Then the game can be represented by the bimatrix

$$\begin{pmatrix} (a_{11}, b_{11}) & \dots & (a_{1m}, b_{1m}) \\ \dots & \dots & \dots \\ (a_{n1}, b_{n1}) & \dots & (a_{nm}, b_{nm}) \end{pmatrix}$$

where a_{ij} (b_{ij}) is the utility of the row (column)player when row plays strategy i and column strategy j.

Denote by (A, B) such a game.

Finite games

Corollary

A finite game (A, B) admits always a Nash equilibrium profile in mixed strategies

In this case X and Y are simplexes, while $f(x, y) = x^t Ay, g(x, y) = x^t By$

and thus the assumption of the theorem are fulfilled.

Developing the row by column product we get:

$$f(x, y) = \sum_{i=1,\dots,n, j=1,\dots,m} x_i y_j a_{ij}, \quad g(x, y) = \sum_{i=1,\dots,n, j=1,\dots,m} x_i y_j b_{ij}$$

Finding Nash equilibria: an example

The game:

$$\begin{pmatrix} (1,0) & (0,3) \\ (0,2) & (1,0) \end{pmatrix}$$

Pl1 playing $(p, 1-p)$, PL2 playing $(q, 1-q)$:

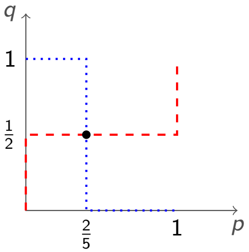
$$f(p,q) = pq + (1-p)(1-q) = p(2q-1) - q + 1$$

$$g(p,q) = 3p(1-q) + 2(1-p)q = q(2-5p) + 3p$$

The best reply multifunctions

$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad 0 \le q \le \frac{1}{2} \\ p \in [0,1] & \text{if} \quad q = \frac{1}{2} \\ p = 1 & \text{if} \quad q > \frac{1}{2} \end{cases}$$

$$BR_2(p) = \begin{cases} q = 1 & \text{if} \quad 0 \le p \le \frac{2}{5} \\ q \in [0,1] & \text{if} \quad p = \frac{2}{5} \\ q = 0 & \text{if} \quad p > \frac{2}{5} \end{cases}$$



Best reaction in pure strategies

The following remark is fundamental to efficiently find Nash Equilibria in finite games

Remark

Once fixed the strategies of the other players, the utility function of one player is linear in its own variable

Thus for every (mixed) strategy y of the second player, $BR_1(y)$ contains at least a pure strategy, since PL1 maximizes a linear function over a simplex. And the same holds for Pl2

Linear equalities and inequalities:Player 1

Remark

Suppose $(\bar{x}, \bar{y})$ *is a NE in mixed strategies. Suppose spt* $\bar{x} = \{1, \dots, k\}$[1], *spt* $\bar{y} = \{1, \dots, l\}$, *and* $f(\bar{x}, \bar{y}) = v$. *Then it holds:*

$$\begin{cases} a_{11}\bar{y}_1 + a_{12}\bar{y}_2 + \cdots + a_{1l}\bar{y}_l & = v \\ \dots & = v \\ a_{k1}\bar{y}_1 + a_{k2}\bar{y}_2 + \cdots + a_{kl}\bar{y}_l & = v \\ a_{(k+1)1}\bar{y}_1 + a_{(k+1)2}\bar{y}_2 + \cdots + a_{(k+1)l}\bar{y}_l & \leq v \\ \dots & \leq v \\ a_{n1}\bar{y}_1 + a_{n2}\bar{y}_2 + \cdots + a_{nl}\bar{y}_l & \leq v \end{cases}$$

The above relations are due the fact that rows used with positive probability must be all optimal (and thus they all give the same expected value), while the other ones are suboptimal

[1] spt $\bar{x} = \{i : \bar{x}_i > 0\}$

Linear equalities and inequalities:Player 2

Remark

Suppose $(\bar{x}, \bar{y})$ is a NE in mixed strategies. Suppose spt $\bar{x} = \{1, \dots, k\}$, spt $\bar{y} = \{1, \dots, l\}$, and $f(\bar{x}, \bar{y}) = v$. Then it holds:

$$\begin{cases} b_{11}\bar{x}_1 + b_{21}\bar{x}_2 + \cdots + b_{k1}\bar{x}_k & = v \\ \dots & = v \\ b_{1l}\bar{x}_1 + b_{2l}\bar{x}_2 + \cdots + b_{kl}\bar{x}_k & = v \\ b_{1(l+1)}\bar{x}_1 + b_{2(l+1)}\bar{x}_2 + \cdots + b_{k(l+1)}\bar{x}_k & \leq v \\ \dots & \leq v \\ b_{1n}\bar{x}_1 + b_{2n}\bar{x}_2 + \cdots + b_{kn}\bar{x}_k & \leq v \end{cases}$$

An example

In the following game, find a, b such that there is a Nash equilibrium with support the first two rows for the first player and the columns 2 and 3 for the second

$$\begin{pmatrix} (2,2) & (a,3) & (3,3) \\ (4,0) & (3,4) & (5,b) \\ (2,3) & (5,2) & (4,26) \end{pmatrix},$$

The system to impose, about the first player:

$$aq + 3 - 3q = 3q + 5 - 5q, \qquad 3q + 5 - 5q \geq 5q + 4 - 4q$$

providing the conditions

$$q = \frac{2}{a-1}, \qquad q \leq \frac{1}{3}.$$

For consistency, this implies $a \geq 7$. For the second player the first column is strictly dominated, and it must be $b = 4$ (otherwise one column dominates the other one) and in this case every $p \in (0,1)$ works.

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Full support

The above system of equalities/inequalities simplifies if one looks for fully mixed[2] Nash equilibria.

Suppose $(\bar{x}, \bar{y})$ is such a Nash equilibrium profile. Then it holds that

$$a_{i1}\bar{y}_1 + a_{i2}\bar{y}_2 + \cdots + a_{im}\bar{y}_m = a_{k1}\bar{y}_1 + a_{k2}\bar{y}_2 + \cdots + a_{km}\bar{y}_m$$

for all $i, k = 1, \dots n$, and similarly

$$b_{1r}\bar{x}_1 + b_{2r}\bar{x}_2 + \cdots + b_{nr}\bar{x}_n = b_{1s}\bar{x}_1 + b_{2s}\bar{x}_2 + \cdots + b_{ns}\bar{x}_n$$

for all $r, s = 1, \dots m$, together with the conditions $p_j, q_j \geq 0, \sum p_i = 1, \sum q_j = 1$

In this case we speak about Indifference principle.

[2] This means that all rows/columns are played with positive probabilities

Brute force algorithm

In general a way to proceed is

1. Guess the supports of the equilibria $spt(\bar{x})$ and $spt(\bar{y})$
2. Ignore the inequalities and find x, y, v, w by solving the linear system of $n + m + 2$ equations

$$\begin{cases} \sum_{i=1}^{n} x_i = 1 & \\ \sum_{j=1}^{m} a_{ij} y_j = v & \text{for all } i \in spt(\bar{x}) \\ x_i = 0 & \text{for all } i \notin spt(\bar{x}) \end{cases}$$

$$\begin{cases} \sum_{j=1}^{m} y_j = 1 & \\ \sum_{i=1}^{n} b_{ij} x_i = w & \text{for all } j \in spt(\bar{y}) \\ y_j = 0 & \text{for all } j \notin spt(\bar{y}) \end{cases}$$

3. Check whether the ignored inequalities are satisfied.
   If $x_i \geq 0, y_j \geq 0, \sum_{j=1}^{m} a_{ij} y_j \leq v$ and $\sum_{i=1}^{n} b_{ij} x_i \leq w$ then Stop: we have found a mixed equilibrium profile. Otherwise, go back to step 1 and try another guess of the supports.

Lemke-Howson Algorithm

Enumerating all the possible supports in the brute force algorithm quickly becomes computationally prohibitive: there are potentially $(2^n - 1)(2^m - 1)$ options!

For $n \times n$ games the number of combinations grow very quickly

n	# of potential supports
2	9
3	49
4	225
5	961
10	1.046.529
20	1.099.509.530.625

Lemke-Howson proposed a more efficient algorithm... though still with exponential running time in the worst case.

First example: the Braess paradox

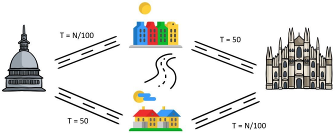

Figure: Commuting

4.000 people travel form one city to another one. Every player wants to minimize time. N is the number of people driving in the corresponding road

What are the Nash equilibria? What happens if the North-South street between the two small cities is made available to cars and time to travel on it is 5 minutes?

El Farol bar

In Santa Fe there are 500 young people, happy to go to the El Farol bar. More people in the bar, happier they are, till they reach 300 people. They can also choose to stay at home. So utility function can be assumed to be 0 if they stay at home, $u(x) = x$ if $x \leq 300$, $u(x) = 300 - x$ if $x > 300$.

In all pure Nash Equilibria the outcome is 300 people in the bar, 200 at home. This creates a great dissymmetry among players

In experimental economics it is often observed in similar situations that players adapt to a symmetric mixed Nash equilibrium profile.

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Duopoly models

Two firms choose quantities of a good to produce. Firm 1 produces quantity q_1, firm 2 produces quantity q_2, the unitary cost of the good is $c > 0$ for both firms. A quantity $a > c$ of the good saturates the market. The price $p(q_1, q_2)$ is

$$p = \max\{a - (q_1 + q_2), 0\}$$

Payoffs:

$$u_1(q_1, q_2) = q_1 p(q_1, q_2) - cq_1 = q_1(a - (q_1 + q_2)) - cq_1,$$
$$u_2(q_1, q_2) = q_2 p(q_1, q_2) - cq_2 = q_2(a - (q_1 + q_2)) - cq_2.$$

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The monopolist

Suppose $q_2 = 0$.

Firm 1 maximizes $u(q_1) = q_1(a - q_1) - cq_1$.

$$q_M = \frac{a-c}{2}, \quad p_M = \frac{a+c}{2} \quad u_M(q_M) = \frac{(a-c)^2}{4}$$

The duopoly

The utility functions are strictly concave and non positive at the endpoints of the domain, thus the first derivative must vanish:

$$a - 2q_1 - q_2 - c = 0, \qquad a - 2q_2 - q_1 - c = 0,$$

$$q_i = \frac{a-c}{3}, \quad p = \frac{a+2c}{3} \quad u_i(q_i) = \frac{(a-c)^2}{9}.$$

The case with a leader

One firm, the Leader, announces its strategy, and the other one, the Follower, acts taking for granted the announced strategy of the Leader.

$$\bar{q}_2(q_1) = \frac{a - q_1 - c}{2}.$$

The Leader maximizes

$$u_1(q_1, \frac{a - q_1 - c}{2})$$

$$\bar{q}_1 = \frac{a-c}{2},\ \bar{q}_2 = \frac{a-c}{4},\ u_1(\bar{q}_1, \bar{q}_2) = \frac{(a-c)^2}{8},\ u_2(\bar{q}_1, \bar{q}_2) = \frac{(a-c)^2}{16}$$

$$p = \frac{a+3c}{4}$$

Comparing the three cases

Monopoly

$$q_M = \frac{a-c}{2}, \quad p_M = \frac{a+c}{2} \quad u_M(q_M) = \frac{(a-c)^2}{4}$$

Duopoly

$$q_i = \frac{a-c}{3}, \quad p = \frac{a+2c}{3} \quad u_i(q_i) = \frac{(a-c)^2}{9}.$$

Leader

$$\bar{q}_1 = \frac{a-c}{2},\ \bar{q}_2 = \frac{a-c}{4},\ u_1(\bar{q}_1, \bar{q}_2) = \frac{(a-c)^2}{8},\ u_2(\bar{q}_1, \bar{q}_2) = \frac{(a-c)^2}{16}$$

$$p = \frac{a+3c}{4}$$

Summarizing

Making a comparison with the case of a monopoly, we see that:

- the price is lower in the duopoly case;
- the total quantity of product in the market is superior in the duopoly case;
- the total payoff of the two firms is less than the payoff of the monopolist.

In particular, the two firm could consider the strategy of equally sharing the payoff of the monopolist, but this is not a NE profile! The result shows a very reasonable fact, the consumers are better off if there is no monopoly.

Finding the Cournot equilibrium by eliminating strictly dominated strategies: proof by Gianpaolo Di Pietro (student)

Given the utilities:

$$u_1(q_1, q2) = -q_1^2 + (a - c - q_2)q_1 \quad (1)$$

$$u_2(q_1, q2) = -q_2^2 + (a - c - q_1)q_2 \quad (2)$$

$$q_1, q_2 \in [0, a]$$

$$-2q_1 + a - c - q_2 \leq 0 \implies q_1 \geq \tfrac{a-c-q_2}{2}; \text{ since } q_2 \geq 0$$

$$q_1 \leq \tfrac{a-c}{2} \text{ is strictly dominated by } \tfrac{a-c}{2}.$$

Since the two utility functions are symmetric, $\frac{a-c}{2}$ dominates all larger quantities for q_2.

Continued

In a similar way (setting the derivative to be positive for all quantities of the opponent), it is seen that $\frac{a-c}{4}$ strictly dominates all smaller quantities q_1, q_2

Now proceed iteratively
Suppose $q_1, q_2 \in [l_i, u_i]$. We have that:

$$\begin{cases} l_i = \frac{a-c-u_i}{2} \\ u_{i+1} = \frac{a-c-l_i}{2} \end{cases} \implies \begin{cases} u_{i+1} = \frac{a-c}{4} + \frac{u_i}{4} \\ l_{i+1} = \frac{a-c}{4} + \frac{l_i}{4} \end{cases}$$

where $l_0 = 0, u_0 = a$.

Now, we seek for the fixed points of the function $f(u) = \frac{a-c-\frac{a-c-u}{2}}{2}$

Conclusion

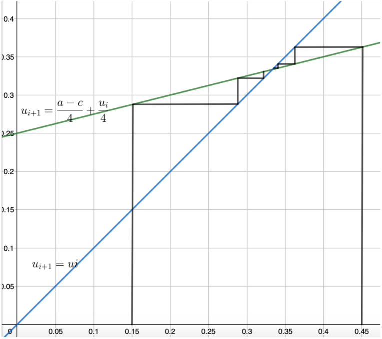


Figure: Graphical proof for the difference equation convergence

Finite games with common payoffs

Consider a finite game with strategy sets X_i and suppose that all the players have the same payoff $p : X \to \mathbb{R}$, that is

$$u_i(x_1, \dots, x_n) = p(x_1, \dots, x_n).$$

Take $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \in X$ a strategy profile such that $p(\bar{x}) \geq p(x)$ for all strategy profiles $x \in X$.
Then $\bar{x}$ is a Nash equilibrium in pure strategies.

Remark

There might be other Nash equilibria in pure or mixed strategies. However, playing $\bar{x}$ is the best that every player could ever hope for.

Best response dynamics

Consider the following payoff-improving procedure:

1. Start from an arbitrary strategy profile $(x_1, \ldots, x_n) \in X$
2. Ask if any player has a better strategy x_i' that strictly increases her payoff

$$u_i(x_i', x_{-i}) > u_i(x_i, x_{-i})$$

   - If yes, replace x_i with x_i' and repeat.
   - Otherwise stop: we have found a pure Nash equilibrium profile!

Each iteration strictly increases the value $p(x)$ so that no strategy profile $x \in X$ can be visited twice. Since X is a finite set, the procedure must reach a pure Nash equilibrium after at most $|X|$ steps.

Observe: this procedure guarantees to reach the global maximum $\bar{x}$

Payoff equivalence

Consider now a general finite game with payoffs $u_i : X \to \mathbb{R}$. How do best responses and Nash equilibria change if we add a constant c_i to the payoff of player i?

$$\tilde{u}_i(x_1, \dots, x_n) = u_i(x_1, \dots, x_n) + c_i$$

What if c_i is not constant but it depends only on x_{-i} and not on x_i?

Best responses and equilibria remain the same!

The payoffs $\tilde{u}_i$ and u_i are said *diff-equivalent* for player i if the difference

$$\tilde{u}_i(x_1, \dots, x_n) - u_i(x_1, \dots, x_n) = c_i(x_{-i})$$

does not depend on her decision x_i but only on the strategies of the other players.

Payoff equivalence

By definition, diff-equivalent payoffs are such that for all $x_i', x_i \in X_i$

$$\tilde{u}_i(x_i', x_{-i}) - u_i(x_i', x_{-i}) = \tilde{u}_i(x_i, x_{-i}) - u_i(x_i, x_{-i}).$$

Denoting $\Delta f(x_i', x_i, x_{-i}) = f(x_i', x_{-i}) - f(x_i, x_{-i})$ this can be rewritten as

$$\Delta \tilde{u}_i(x_i', x_i, x_{-i}) = \Delta u_i(x_i', x_i, x_{-i}). \tag{3}$$

Theorem

Finite games with diff-equivalent payoffs have the same pure Nash equilibria.

Proof The best reaction multifunction, for every player i, is the same when considering two diff-equivalent payoffs. ∎

We use here payoffs diff-equivalent, but it is possible to consider different equivalences: what matters is to maintain unchanged the Best Reaction multifunctions.

Potential games

Definition

A finite game with strategy sets X_i and payoffs $u_i : X \to \mathbb{R}$ is called a potential game if it is diff-equivalent to a game with common payoffs, that is, there exists a potential function $p : X \to \mathbb{R}$ such that for each i, for every $x_{-i} \in X_{-i}$, and all $x_i', x_i \in X_i$ we have

$$\Delta u_i(x_i', x_i, x_{-i}) = \Delta p(x_i', x_i, x_{-i}).$$

Corollary

1. *Every finite potential game has at least one pure Nash equilibrium*
2. *In a finite potential game every best response iteration reaches a pure Nash equilibrium in finitely many steps.*

A toy example

$$\begin{pmatrix} (10,10) & (0,11) \\ (11,0) & (1,1) \end{pmatrix}$$

A potential

$$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

For Player 2

- Differences when the first row is fixed: $11 - 10 = 1 - 0$
- Differences when the second row is fixed: $1 - 0 = 2 - 1$

For Player 1

- Differences when the first column is fixed: $11 - 10 = 1 - 0$
- Differences when the second column is fixed: $1 - 0 = 2 - 1$

How to find a potential

A potential $p : X \to \mathbb{R}$ is characterized by

$$\Delta p(x_i', x_i, x_{-i}) = \Delta u_i(x_i', x_i, x_{-i}).$$

Adding a constant to $p(\cdot)$ provides a new potential.
Fix an arbitrary profile $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$ and set $p(\bar{x}) = 0$.
Now the potential $p(\cdot)$ is determined uniquely:

$$\begin{array}{rcl}
p(x_1, x_2, \dots, x_n) - p(\bar{x}_1, x_2, \dots, x_n) & = & u_1(x_1, x_2, \dots, x_n) - u_1(\bar{x}_1, x_2, \dots, x_n) \\
p(\bar{x}_1, x_2, \dots, x_n) - p(\bar{x}_1, \bar{x}_2, \dots, x_n) & = & u_2(\bar{x}_1, x_2, \dots, x_n) - u_2(\bar{x}_1, \bar{x}_2, \dots, x_n) \\
 & \vdots & \\
p(\bar{x}_1, \dots, \bar{x}_{n-1}, x_n) - p(\bar{x}_1, \dots, \bar{x}_{n-1}, \bar{x}_n) & = & u_n(\bar{x}_1, \dots, \bar{x}_{n-1}, x_n) - u_n(\bar{x}_1, \dots, \bar{x}_{n-1}, \bar{x}_n) \\
\hline
\end{array}$$

$$\Rightarrow \quad p(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} [u_i(\bar{x}_1 \dots \bar{x}_{i-1}, x_i \dots x_n) - u_i(\bar{x}_1 \dots \bar{x}_{i-1}, \bar{x}_i \dots x_n)]$$

Existence of a potential

If the game admits a potential the sum on the right hand side of the previous slide is independent of the particular order used.

The converse is also true. However, checking that all these orders yield the same answer is impractical for more than 2 or 3 players.

Example: computing a potential

Check that the following is a potential game

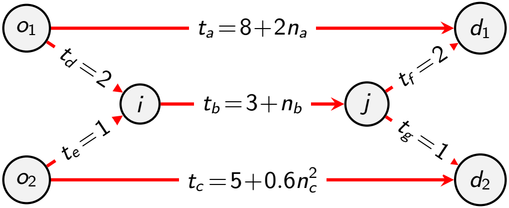
$$\begin{pmatrix} (2,5) & (2,6) & (3,7) & (8,9) & (5,7) \\ (1,4) & (1,5) & (3,7) & (2,3) & (0,2) \\ (6,5) & (2,2) & (0,0) & (6,3) & (3,1) \end{pmatrix}$$

Potential:

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 2 \\ -1 & 0 & 2 & -2 & -3 \\ 4 & 1 & -1 & 2 & 0 \end{pmatrix}$$

Example 1: Routing games

Consider n drivers traveling between different origins and destinations in a city. The transport network is modeled as a graph (N, A) with node set N and arcs A. Because of congestion, the travel time of an arc $a \in A$ is a non-negative increasing function $t_a = t_a(n_a)$ of the load $n_a = \#$ of drivers using the arc. We set $t_a(0) = 0$.



One pure strategy for i is a route $r_i = a_1 a_2 \cdots a_\ell$, that is, a sequence of arcs connecting her origin $o_i \in N$ to her destination $d_i \in N$. Her total travel time is

$$u_i(r_1, \ldots, r_n) = \sum_{a \in r_i} t_a(n_a) \quad ; \quad n_a = \#\{j : a \in r_j\}$$

Here u_i represents a cost for Player i.

Example 1: Routing games

To minimize travel time, drivers may restrict to simple paths with no cycles: nodes are visited at most once. Hence, the strategy set for player i is the set X_i of all simple paths connecting o_i to d_i.

Theorem (Rosenthal'73)

A routing game admits the potential

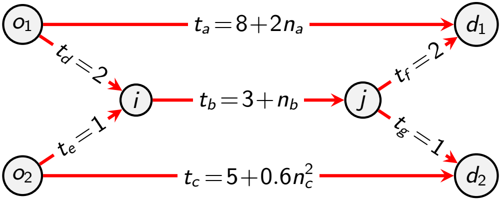
$$p(r_1,\ldots,r_n)=\sum_{a\in A}\sum_{k=0}^{n_a} t_a(k) \quad ; \quad n_a=\#\{j\colon a\in r_j\}.$$

Proof It suffices to note that for $r=(r_1,\ldots,r_n)$ we have

$$p(r)-u_i(r)=\sum_{a\in A}\sum_{k=0}^{n_a} t_a(k)-\sum_{a\in r_i} t_a(n_a)=\sum_{a\in A}\sum_{k=1}^{n_a^{-i}} t_a(k)$$

where $n_a^{-i}=\#\{j\neq i\colon a\in r_j\}$ is the number of drivers other than i using arc a. Hence, the difference $p(r)-u_i(r)$ depends only on r_{-i} and not on r_i. ■

Example revisited



Two players go from O_1 to d_1 and one from O_2 to d_2. $r_1 = a$, $r_2 = dbf$, $r_3 = ebg$.

$\sum_{k=1}^{n_a} t_a(k)$ for every arc, under the profile r:

1. a 10
2. b $4 + 5$
3. c 0
4. d 2
5. e 1
6. f 2
7. g 1

Costs:

1. for player 1 = 10 (arc a)
2. for player 2 = 2 (arc d)+ 5 (arc b)+ 2 (arc f)
3. for player 3 = 1 (arc d)+ 5 (arc b)+ 1 (arc g)

Difference $p(r_1, r_2, r_3) - u_1(r_1, r_2, r_3)$ depends only from r_2, r_3 and the same for the other players.

Example 2: Congestion games

A routing game is a special case of the more general class of *Congestion games*. Here each player $i = 1, \ldots, n$ has to perform a certain task which requires some resources taken from a set R. The strategy set X_i for player i contains all subsets $x_i \subseteq R$ that allow her to perform the task.

Each resource $r \in R$ has a cost $c_r(n_r)$ which depends on the number of players that use the resource. Player i only pays for the resources she uses

$$u_i(x_1, \ldots, x_n) = \sum_{r \in x_i} c_r(n_r) \quad ; \quad n_r = \#\{j : r \in x_j\}.$$

Verify that $p(x_1, \ldots, x_n) = \sum_{r \in R} \sum_{k=1}^{n_r} c_r(k)$ is a potential.

Observe: here u_i represents a cost for Player i

Example 3: Network connection games

A telecommunication network (N, A) is under construction. Each player i wants a route r_i to be built between a certain origin o_i and a destination d_i. The cost v_a of building an arc $a \in A$ is shared evenly among the players who use it.

Hence, the cost for player i is

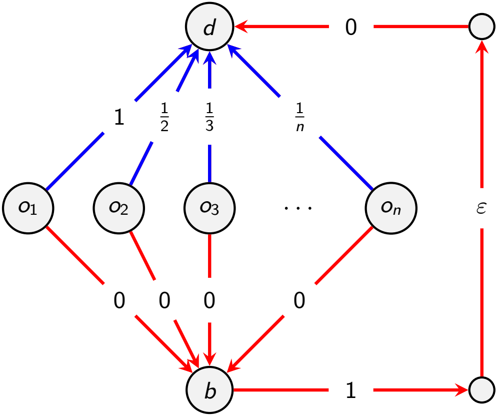
$$u_i(r_1, \dots, r_n) = \sum_{a \in r_i} \frac{v_a}{n_a} \quad ; \quad n_a = \#\{j : a \in r_j\}.$$

Contrary to the congestion game, in this case there is an incentive to use congested arcs as this reduces the cost.

This is again a congestion game with potential

$$p(r_1, \dots, r_n) = \sum_{a \in A : n_a > 0} v_a \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + \tfrac{1}{n_a}\right).$$

Example 3: Network connection games



Social cost and efficiency

Nash equilibria need not be Pareto efficient and can be bad for all the players as in the Braess' paradox, the Prisoner's dilemma, or the Tragedy of the commons.
An important question is to quantify how bad can be the outcome of a game.
To answer this question it is necessary to define what is good and what is bad.
Different choices are possible. We assume from now on that, like in most previous examples, costs, rather than utilities, of the players are given.
The quality of a strategy profile $x = (x_1, \ldots, x_n)$ is measured through a social cost function $x \mapsto C(x)$ where $C : X \to \mathbb{R}_+$. The smaller $C(x)$ the better the outcome $x \in X$. The benchmark is the minimal value that a benevolent social planner could achieve

$$Opt = \min_{x \in X} C(x).$$

For $x \in X$ the quotient $\frac{C(x)}{Opt}$ measures how far is x from being optimal. A large value implies a big loss in social welfare, a quotient close to 1 implies that x is almost as efficient as an optimal solution.

Price-of-Anarchy and Price-of-Stability

Definition

Let $NE \subseteq X$ be the set of pure Nash equilibria of a cost game. The Price-of-Anarchy and the Price-of-Stability are defined respectively by

$$PoA = \max_{\bar{x} \in NE} \frac{C(\bar{x})}{Opt} \qquad ; \qquad PoS = \min_{\bar{x} \in NE} \frac{C(\bar{x})}{Opt}$$

$1 \leq PoS \leq PoA$

- $PoA \leq \alpha$ means that in every possible pure equilibrium the social cost $C(\bar{x})$ is no worse than $\alpha\, Opt$
- $PoS \leq \alpha$ means that there exists some equilibrium with social cost at most $\alpha\, Opt$.

Social cost

A natural cost function aggregates the costs of all the players

$$C(a) = \sum_{i=1}^{n} u_i(a)$$

Example

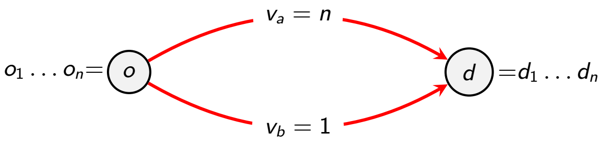
- *In the routing game the function is the total time traveled by all the players*

$$C(r_1, \ldots, r_n) = \sum_{x \in X} n_x t_x(n_x) \quad ; \quad n_x = \#\{j : x \in r_j\}.$$

- *In the network connection game the function gives the total investment required to connect all the players*

$$C(r_1, \ldots, r_n) = \sum_{x \in X : n_x > 0} v_x \quad ; \quad n_x = \#\{j : x \in r_j\}.$$

Example: PoA and PoS — Network connection game



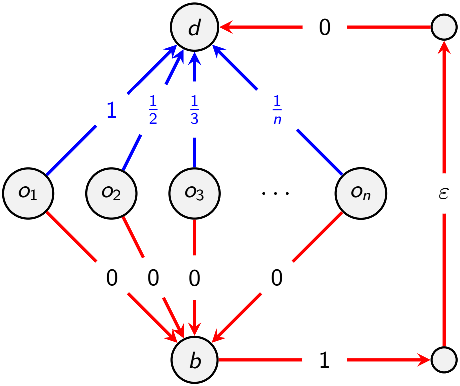
There are two Nash equilibria: either all go above or below. The sum of the costs going above is n, below is 1. Thus

$$Opt = 1$$

$$PoS = 1$$

$$PoA = n \to \infty$$

Example: PoA and PoS — Network connection game



$$Opt = 1 + \varepsilon$$

$$C(\bar{x}) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = H_n$$

$$PoA = PoS = \frac{H_n}{1+\varepsilon} \sim \ln(n) \to \infty$$

Verify that the unique Nash equilibrium profile $\bar{x}$ is such that the strategy of each player is to directly connect to the destination and it can be obtained by elimination of strictly dominated strategies

An estimate for PoS

Proposition

Consider a cost minimization finite potential game with potential $p : X \to \mathbb{R}$, and suppose that there exist $\alpha, \beta > 0$ such that

$$\frac{1}{\alpha} C(x) \le p(x) \le \beta C(x) \quad \forall\, x \in X.$$

Then PoS $\le \alpha\beta$.

Proof Let $\bar{x}$ be a minimum of $p(\cdot)$ so that $\bar{x}$ is a Nash equilibrium. For all $x \in X$

$$\frac{1}{\alpha} C(\bar{x}) \le p(\bar{x}) \le p(x) \le \beta C(x)$$

Since this is true for all x, then $C(\bar{x}) \le \alpha\beta\, Opt$. ■

Application: PoS in network connection games

Proposition

Consider a network congestion game with n players on a general graph (N, X) with arc construction costs $v_x \geq 0$. Then $PoS \leq H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

Proof In this case the potential and the social cost are

$$p(r_1, \ldots, r_n) = \sum_{x \in X} \sum_{k=1}^{n_x} \frac{v_x}{k}$$

$$C(r_1, \ldots, r_n) = \sum_{x \in X : n_x > 0} v_x$$

so that $C(r) \leq p(r) \leq H_n C(r)$ and the previous result yields $PoS \leq H_n$.
■

A final remark

In case a game deals with utilities rather than costs, one defines

$$Opt = \max_{x \in X} U(x)$$

where $U(x)$ is some fixed social utility function.

Definition

Let $NE \subseteq X$ be the set of pure Nash equilibria of the game. The Price-of-Anarchy and the Price-of-Stability for a utility game are defined respectively by

$$PoA = \max_{\bar{x} \in NE} \frac{Opt}{U(\bar{x})} = \frac{Opt}{\min_{\bar{x} \in NE}(U(\bar{x}))} \quad PoS = \min_{\bar{x} \in NE} \frac{Opt}{U(\bar{x})} = \frac{Opt}{\max_{\bar{x} \in NE}(U(\bar{x}))}$$

This is to have that high prices PoS and PoA continue to indicate games with bad behavior of Nash equilibrium profiles.