Zero sum games

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Summary of the slides

1. Zero sum game in strategic form
2. Conservative values of the players
3. Optimal strategies for the players and common value
4. $v_1 \leq v_2$ for arbitrary games
5. Mixed extension of the (finite) zero sum game
6. The von Neumann theorem
7. Basics of convexity
8. A separation theorem
9. Proof of the von Neumann theorem
10. Finding optimal strategies for the players as LP problems
11. Some basics of Linear Programming
12. Duality: the weak and the strong duality theorems
13. Complementarity conditions
14. Equivalent formulations for finding optimal strategies
15. Complementarity conditions in the zero sum games
16. Fair games

General form

Definition

A two player zero sum game in strategic form is the triplet $(X, Y, f : X \times Y \to \mathbb{R})$

X is the strategy space of Pl1, Y the strategy space of Pl2, $f(x, y)$ is what Pl1 gets from Pl2, when they play x, y respectively. Thus f is the utility function of Pl1, while for Pl2 the utility function g is $g = -f$.

Finite game

In the finite case $X = \{1, 2, \dots, n\}$, $Y = \{1, 2, \dots, m\}$ the game is described by a payoff matrix P

Example

$$P = \begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}$$

Pl1 selects row i, Pl2 selects column j.
In general

$$\begin{pmatrix} p_{11} & \dots & p_{1m} \\ \dots & \dots & \dots \\ p_{n1} & \dots & p_{nm} \end{pmatrix}$$

where p_{ij} is the payment of Pl2 to Pl1 when they play i, j respectively.

How to solve them

Consider the game

$$\begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}$$

- $\min_j p_{1j} = 1$, $\min_j p_{2j} = 5$, $\min_j p_{3j} = 0 \qquad v_1 = 5$
- $\max_i p_{i1} = 8$, $\max_i p_{i2} = 5$, $\max_i p_{i3} = 8, \qquad v_2 = 5$

Thus

- Pl1 can guarantee herself to get at least

$$v_1 = \max_i \min_j p_{ij}$$

- Pl2 can guarantee himself to pay no more than

$$v_2 = \min_j \max_i p_{ij}$$

In the example $v_1 = v_2 = 5$ and
Rational outcome 5.Rational behavior ($\bar{\imath} = 2, \bar{\jmath} = 2$)

Idea of rationality in zero sum games

Suppose

- $v_1 = v_2 := v$,
- $\bar{\imath}$ the row such that $p_{\bar{\imath}j} \geq v_2 = v$ for all j
- $(\bar{\jmath})$ the column such that $p_{i\bar{\jmath}} \leq v_1 = v$ for all i

Then $p_{\bar{\imath}\bar{\jmath}} = v$ and $p_{\bar{\imath}\bar{\jmath}} = v$ is the rational outcome of the game

Remark

- $\bar{\imath}$ *is an optimal strategy for Pl1, because she cannot get more than* v_2*, since* v_2 *is the conservative value of the second player*
- $\bar{\jmath}$ *is an optimal strategy for Pl2, because he cannot pay less than* v_1*, since* v_1 *is the conservative value of the first player*

Remark

Observe $\bar{\imath}$ *maximizes the function* $\alpha(i) = \min_j p_{ij}$, $\bar{\jmath}$ *minimizes the function* $\beta(j) = \max_i p_{ij}$

For arbitrary games

$(X, Y, f : X \times Y \to \mathbb{R})$

The players can guarantee to themselves (almost):

Pl1: $v_1 = \sup_x \inf_y f(x, y)$

PL2: $v_2 = \inf_y \sup_x f(x, y)$

v_1, v_2 are the conservative values of the players

If $v_1 = v_2$, we set $v = v_1 = v_2$ and we say that the game has value v

Optimality

Suppose

1. $v_1 = v_2 := v$
2. there exists strategy $\bar{x}$ such that $f(\bar{x}, y) \geq v$ for all $y \in Y$
3. there exists strategy $\bar{y}$ such that $f(x, \bar{y}) \leq v$ for all $x \in X$

Then

- v is the rational outcome of the game
- $\bar{x}$ is an optimal strategy for Pl1
- $\bar{y}$ is an optimal strategy for Pl2

Observe

- $\bar{x}$ is optimal for Pl1 since it maximizes the function $\alpha(x) = \inf_y f(x, y)$
- $\bar{y}$ is optimal for Pl2 since it minimizes the function $\beta(y) = \sup_x f(x, y)$

Observe: $\alpha(x)$ is the value of the optimal choice of Pl2 if he knows that Pl1 plays x and symmetrically for $\beta(y)$

$v_1 \leq v_2$

Proposition

Let X, Y be nonempty sets and let $f : X \times Y \to \mathbb{R}$ be an arbitrary real valued function. Then

$$\sup_x \inf_y f(x,y) \leq \inf_y \sup_x f(x,y)$$

Proof Observe that, for all x, y,

$$\inf_y f(x,y) \leq f(x,y) \leq \sup_x f(x,y)$$

Thus

$$\alpha(x) = \inf_y f(x,y) \leq \sup_x f(x,y) = \beta(y)$$

Since for all $x \in X$ and $y \in Y$ it holds

$$\alpha(x) \leq \beta(y)$$

it follows

$$\sup_x \alpha(x) \leq \inf_y \beta(y) \quad \blacksquare$$

As a consequence, in every game $v_1 \leq v_2$

Equality need not hold

Example

$$P = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

$v_1 = -1$, $v_2 = 1$

Nothing unexpected...

Case $v_1 < v_2$

Finite case: mixed strategies. Game: $n \times m$ matrix P.

Strategy spaces:

$$\Sigma_n = \{x = (x_1, \dots, x_k) : x_i \geq 0, \sum_{i=1}^{k} x_i = 1\}$$

where $k = n$ for Pl1, $k = m$ for Pl2

$$f(x, y) = \sum_{i=1,\dots,n, j=1,\dots,m} x_i y_j p_{ij} = x^t P y$$

The mixed extension of the initial game P: $(\Sigma_n, \Sigma_m, f(x, y) = x^t P y)$

To prove existence of a rational outcome

To have existence of a rational outcome for the game, need to prove:

1. $v_1 = v_2$ (the two conservative values agree)
2. there exists $\bar{x}$ fulfilling
$$v_1 = \inf_y f(\bar{x}, y)$$
($\bar{x}$ is optimal for Pl1)
3. there exists $\bar{y}$ fulfilling
$$v_2 = \sup_x f(x, \bar{y})$$
($\bar{y}$ is optimal for Pl2)

In the finite case optimal $\bar{x}$ and $\bar{y}$ always exist; thus existence is equivalent to coincidence of the conservative values

The von Neumann theorem

Theorem

A two player, finite, zero sum game as described by a payoff matrix P has a rational outcome

Convexity (1)

Definition

A set $C \subset \mathbb{R}^n$ is said to be convex provided $x, y \in C$, $\lambda \in [0,1]$ imply:

$$\lambda x + (1-\lambda)y \in C$$

Remark

- *The intersection of an arbitrary family of convex sets is convex*
- *A closed convex set with nonempty interior coincides with the closure of its internal points*

Definition

We shall call a convex combination of elements $x_1, \dots, x_n$ any vector x of the form

$$x = \lambda_1 x_1 + \cdots + \lambda_n x_n,$$

with $\lambda_1 \geq 0, \dots, \lambda_n \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$

Convexity (2)

Proposition

A set C is convex if and only if for every $\lambda_1 \geq 0, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$, for every $c_1, \ldots, c_n \in C$, for all n, then $\sum_{i=1}^{n} \lambda_i c_i \in C$

If C is not convex, then there is a smallest convex set containing C: it is the intersection of all convex sets containing C

Definition

The convex hull of a set C, denoted by co C, is:

$$\operatorname{co} C \stackrel{\text{def}}{=} \bigcap_{A \in \mathcal{C}} A$$

where $\mathcal{C} = \{A : C \subset A \ \wedge \ A \text{ is convex}\}$

Convexity (3)

Proposition

Given a set C, then

$$\text{co } C = \{\sum_{i=1}^{n} \lambda_i c_i : \ \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, c_i \in C \ \forall i, n \in \mathbb{N}\}$$

The convex hull of any set C is the set made by all convex combinations of points in C.

An interesting case is when C is a finite collection of points, in such a case co C is called a polytope.

Convexity (4)

Theorem

Given a closed convex set C and a point x outside C, there is a unique element $p \in C$ such that, for all $c \in C$

$$||p - x|| \leq ||c - x||$$

p is characterized by

- $p \in C$
- $(x - p)^t(c - p) \leq 0$ *for all* $c \in C$

A first separation result

Theorem

Let C be a convex proper subset of the Euclidean space $\mathbb{R}^l$, let $\bar{x} \in$ cl C^c. Then there is an element $0 \neq x^* \in \mathbb{R}^l$ such that, $\forall c \in C$:

$$x^{*t} c \geq x^{*t} \bar{x}$$

Proof Suppose $\bar{x} \notin$ cl C and call p its projection on cl C. Then $(\bar{x} - p)^t (c - p) \leq 0$ for all $c \in C$. Setting $x^* = p - \bar{x} \neq 0$

$$x^{*t}(c - \bar{x}) \geq \|x^*\|^2$$

implying

$$x^{*t} c \geq x^{*t} \bar{x}$$

$\forall c \in C$. We can choose $\|x^*\| = 1$. If $\bar{x} \in \overline{C} \setminus C$, take a sequence $\{x_n\} \subset C^c$ such that $x_n \to \bar{x}$. From the first step of the proof, find norm one x_n^* such that

$$x_n^{*t} c \geq x_n^{*t} x_n$$

$\forall c \in C$. Thus, possibly passing to a subsequence, we can suppose $x_n^* \to x^*$, where $\|x^*\| = 1$ (so that $x^* \neq 0$). Now take the limit in the above inequality, to get:

$$x^{*t} c \geq x^{*t} \bar{x}$$

$\forall c \in C$ ■

Separating hyperplane

Corollary

Let C be a closed convex set in a Euclidean space, let x be on the boundary of C. Then there is a hyperplane containing x and leaving all of C in one of the halfspaces determined by the hyperplane

The hyperplane whose existence is established in the Corollary is said the be an hyperplane supporting C at x

Corollary

Let C be a closed convex set in a Euclidean space. Then C is the intersection of all halfspaces containing it

The separation result

Theorem

Let A, C be closed convex subsets of $\mathbb{R}^l$ such that $\operatorname{int} A$ *is nonempty and* $\operatorname{int} A \cap C = \emptyset$*. Then there are $0 \neq x^*$ and $b \in \mathbb{R}$ such that, $\forall a \in A$, $\forall c \in C$*

$$x^{*t} a \geq b \geq x^{*t} c$$

Proof Since $0 \in (\text{ int } A - C)^c$, we can apply the previous separation theorem to find $x^* \neq 0$ such that

$$x^{*t} x \geq 0$$

$\forall x \in \text{int } A - C$. Thus:

$$x^{*t} a \geq x^{*t} c$$

$\forall a \in \text{int } A$, $\forall c \in C$. This implies

$$x^{*t} a \geq x^{*t} c$$

$\forall a \in \text{cl int } A = A$, $\forall c \in C$ ■

$H = \{x : x^{*t} x = b\}$ is called the separating hyperplane: A and C are contained in the two different halfspaces generated by H.

Visualizing separation

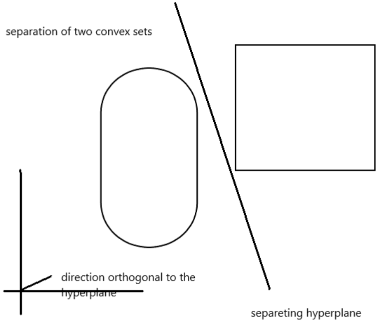

Figura: The line contains elements x such that $\langle x^*, x \rangle = cost$ and divides the plane in two halfplanes one where the inner product il greater than cost, the other one where it is smaller

Optimality in Pure strategies

Theorem

If a player knows the strategy played by the other player, she can always use a pure strategy to get the best outcome.

Proof Consider f.i. the second player, knowing that the first one plays a mixed strategy $\bar{x}$. Then the second player must minimize the function

$$f(\bar{x}, y) = \bar{x}^t P y$$

over the simplex Σ_m. The maximum is reached in at least one vertex e_j. This corresponds to a pure strategy ■

Proving optimality

Given the payoff matrix P, and denoting by $p_{\cdot j}$ (respectively $p_{i\cdot}$ the column j (resp. the row i, seen as a row vector), the above theorem implies that, setting $f(x,y) = x^t P y$, the payoff of the first player in the (mixed extension of the) game, in order to verify the existence of a rational outcome we need to prove existence of $\bar{x}$, $\bar{y}$, v such that

- $\bar{x}^t P e_j = \bar{x}^t p_{\cdot j} \geq v$ for every column j;
- ${e_i}^t p_{i\cdot} \bar{y} \leq v$ for every row i.

The proof of vN theorem (1)

Proof Suppose all entries p_{ij} of the matrix P are positive. Consider the vectors $p_1, \dots, p_m$ of $\mathbb{R}^n$, where p_j denotes the j^{th} column of the matrix P. Call C the convex hull of these vectors, set

$$Q_t = \{x \in \mathbb{R}^n : x_i \leq t\} \ \wedge \ v = \sup\{t \geq 0 : Q_t \cap C = \emptyset\}$$

Since int $Q_v \cap C = \emptyset$, Q_v and C can be separated by an hyperplane: there are coefficients $\bar{x}_1, \dots, \bar{x}_n$, not all zero, and $b \in \mathbb{R}$ such that, for all $u = (u_1, \dots, u_n) \in Q_v$, $w = (w_1, \dots, w_n) \in C$

$$\sum_{i=1}^{n} \bar{x}_i u_i = \langle \bar{x}, u \rangle \leq b \leq \sum_{i=1}^{n} \bar{x}_i w_i = \langle \bar{x}, w \rangle$$

It holds

1. All $\bar{x}_i$ must be nonnegative and, since they cannot be all zero, we can assume $\sum \bar{x}_i = 1$
2. $b = v$; First of all, since $\bar{v} := (v, \dots, v) \in Q_v$, from $\langle \bar{x}, \bar{v} \rangle = v$ we get $b \geq v$. Suppose now $b > v$, and take $a > 0$ so small that $b > v + a$. Then $\sup\{\sum_{i=1}^{n} \bar{x}_i u_i \ : \ u \in Q_{v+a}\} < b$, and this implies $Q_{v+a} \cap C = \emptyset$, against the definition of v
3. $Q_v \cap C \neq \emptyset$. Let $w \in Q_v \cap C$. Then $\bar{w} = \sum_{j=1}^{m} \bar{y}_j p_j$, for some $\Sigma_m \ni \bar{y} = (\bar{y}_1, \dots, \bar{y}_m)$. Observe that, since $\bar{w} \in Q_v$, then $\bar{w}_i \leq v$ for all i.

The proof of vN theorem: conclusion

We now prove that $\bar{x}$ is optimal for the first player, that $\bar{y}$ is optimal for the second player, and that v is the value of the game.

- About the first player: since $\bar{x}^t w \geq v$ for every $w \in C$, by the separation result, and since obviously every column $p_{\cdot j} \in C$, then

$$\bar{x}^t p_{\cdot j} \geq v$$

  for all j;

- Now consider $\sum_{j=1}^{m} \bar{y}_j p_j = \bar{w} \in Q_v \cap C$ as before. Then $w_i = \bar{y} p_{i\cdot}$. Since $w \in Q_v$, then $w_i \leq v$ for every i and

$$v \geq w_i = \bar{y} p_{i\cdot}$$

  This concludes the proof. ■

An example

Consider

$$P = \begin{pmatrix} 7 & 1 & 4 & 9 \\ 3 & 10 & 6 & 2 \\ 4 & 5 & 3 & 0 \end{pmatrix}.$$

The third row is strictly dominated by a convex combination of the first two. Thus the payoff matrix reduces to

$$P' = \begin{pmatrix} 7 & 1 & 4 & 9 \\ 3 & 10 & 6 & 2 \end{pmatrix}.$$

It is easy to show that the line $x + y = 10 = \frac{1}{2}x + \frac{1}{2}y = 5$ contains the first and the third column of P', while the second and fourth column verify $x + y > 10$. Thus $(\frac{1}{2}, \frac{1}{2}, 0)$ is the optimal strategy of the first player and $v = 5$. Moreover, to find the optimal strategy of the second player, we need to write $(5, 5)$ as a convex combination of the vectors of columns 1 and 3. This provides $(\frac{1}{3}, 0, \frac{2}{3}, 0)$ as optimal strategy of the second player.

How to find optimal strategies in the general case

Von Neumann proof can be efficiently used to find rational outcomes of payoff matrices than can be reduced to matrices where one player has only two strategies. This is due to the fact that it is easy to visualize, in this case, the separation line. But in more dimensions this becomes more complicated, since it is not clear when and where the set Q_t meets C. Another method must be used.

Finding optimal strategies:Pl1

Pl1 must choose a probability distribution $\Sigma_n \ni x = (x_1, \dots, x_n)$ in order to maximize v with the constraints:

$$
\begin{aligned}
& x_1 p_{11} + \cdots + x_n p_{n1} \geq v \\
& \dots \\
& x_1 p_{1j} + \cdots + x_n p_{nj} \geq v \\
& \dots \\
& x_1 p_{1m} + \cdots + x_n p_{nm} \geq v
\end{aligned}
$$

Finding optimal strategies:Pl2

Pl2 must choose a probability distribution $\Sigma_m \ni y = (y_1, \dots, y_m)$ in order to minimize w with the constraints:

$$
\begin{aligned}
& y_1 p_{11} + \cdots + y_m p_{1m} \leq w \\
& \dots \\
& y_1 p_{i1} + \cdots + y_m p_{im} \leq w \\
& \dots \\
& y_1 p_{n1} + \cdots + y_m p_{nm} \leq w
\end{aligned}
$$

In matrix form

Pl1:

$$\begin{cases} \max_{x,v} v : \\ P^t x \geq v1_m \\ x \geq 0 \quad 1^t x = 1 \end{cases} \tag{1}$$

Pl2:

$$\begin{cases} \min_{y,w} w : \\ Py \leq w1_n \\ y \geq 0 \quad 1^t y = 1 \end{cases} \tag{2}$$

where 1 is a vector of the right dimension made by all 1's.
(1) and (2) are Linear Programming (LP) problems.

Dual linear programs: Form 1

Definition

The following two linear programs are said to be in duality:

$$(P)\quad \begin{cases} \min c^t x \\ Ax \geq b \\ x \geq 0 \end{cases} \qquad\qquad (D)\quad \begin{cases} \max b^t y \\ A^t y \leq c \\ y \geq 0 \end{cases}$$

The min problem is called primal problem and the max is called dual problem.

Dual linear programs: Form 2

Definition

The following two linear programs are said to be in duality:

$$(P) \quad \begin{cases} \min c^t x \\ Ax \geq b \end{cases} \qquad\qquad (D) \quad \begin{cases} \max b^t y \\ A^t y = c \\ y \geq 0 \end{cases}$$

The minimization problem in the second form can be written in an equivalent way in the first form; dualizing this shows that the dual is equivalent to the dual of the second form.

Feasibility of dual programs

Easy examples show that, given two problems in duality,

- They can be both infeasible
- Only one can be feasible
- Both can be feasible

Example 1

Consider

$$\begin{cases} \min x_1 + x_2 \\ x_1 + 2x_2 \geq 1 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Its dual is

$$\begin{cases} \max y \\ y \leq 1 \\ 2y \leq 1 \\ y \geq 0 \end{cases}$$

Since $(x_1, x_2) = (0, \frac{1}{2})$ fulfills the constraints of the primal problem and $y = \frac{1}{2}$ fulfills the constraints of the dual problem, they are both feasible.

Examples 2,3

Consider

$$\begin{cases} \min x_1 - x_2 \\ x_1 + x_2 \geq 2 \\ -x_1 - x_2 \geq -1 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Its dual is

$$\begin{cases} \max 2y_1 - y_2 \\ y_1 - y_2 \leq 1 \\ y_1 - y_2 \leq -1 \\ y \geq 0 \end{cases}$$

The primal is infeasible while $(0, 1)$ is feasible in the dual.

Taking $A = 0$, $b = (1, \dots, 1)$ and $c = (-1, \dots, -1)$ shows that both problems can be infeasible.

Weak duality theorem

Theorem

Let v be the value of the primal min problem and V the value of the dual max problem. Then

$$v \geq V$$

Proof

Form 1:

$$c^t x \geq (A^t y)^t x = y^t A x \geq y^t b = b^t y$$

Since this is true for all admissible x and y the result follows.

Form 2:

$$c^t x = (A^t y)^t x = y^t A x \geq y^t b = b^t y$$

∎

Strong duality theorem

Theorem

- *If the primal and dual problems are feasible, then both problems have optimal solutions* $\bar{x}, \bar{y}$ *and the optimal values coincide*

$$v = c^t \bar{x} = b^t \bar{y} = V.$$

  In this case we say that there is no duality gap.
- *If the primal is feasible and the dual is infeasible, then* $v = V = -\infty$
- *If the primal is infeasible and the dual is feasible, then* $v = V = +\infty$
- *If both the primal and the dual are infeasible, then* $v = \infty > V = -\infty$

Corollary

If one problem is feasible and has an optimal solution, then also the dual problem is feasible and has solutions. Moreover there is no duality gap.

Complementarity conditions: Form 1

$$(P)\ \begin{cases} \min c^t x \\ Ax \geq b, x \geq 0 \end{cases} \quad ; \quad (D)\ \begin{cases} \max b^t y \\ A^t y \leq c, y \geq 0 \end{cases}$$

Theorem

Let $\bar{x}, \bar{y}$ be primal and dual feasible. Then $\bar{x}, \bar{y}$ are simultaneously optimal iff

$$(CC) \begin{cases} (\forall i = 1, \dots, n) & \bar{x}_i > 0 \Rightarrow \sum_{j=1}^m a_{ji}\bar{y}_j = c_i \\ (\forall j = 1, \dots, m) & \bar{y}_j > 0 \Rightarrow \sum_{i=1}^n a_{ij}\bar{x}_i = b_j \end{cases}$$

Proof Since $c^t x \geq y^t Ax \geq b^t y$ it follows that $\bar{x}, \bar{y}$ are optimal iff

$$c^t \bar{x}^t = \bar{y}^t A\bar{x} = b^t \bar{y}$$

This is equivalent to

$$\bar{x}^t(A^t\bar{y} - c) = 0 \qquad \text{and} \qquad \bar{y}^t(A\bar{x} - b) = 0$$

Since $\bar{x}, \bar{y} \geq 0$ and $A\bar{x} \geq b, A^t\bar{y} \leq c$ the latter are equivalent to (CC). ∎

An example

Consider

$$\begin{cases} \min x_1 + x_2 : \\ 2x_1 + x_2 \geq 2 \\ x_1 + 2x_2 \leq 2 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Its dual is

$$\begin{cases} \max 2y_1 - 2y_2 : \\ 2y_1 - y_2 \leq 1 \\ y_1 - 2y_2 \leq 1 \\ y_1 \geq 0, y_2 \geq 0 \end{cases}$$

We have $v = 1$, $(\bar{x}_1, \bar{x}_2) = (1, 0)$; $V = 1$, $(\bar{y}_1, \bar{y}_2) = (\frac{1}{2}, 0)$.

Check of the complementarity conditions:

$$\bar{y}_1 = \frac{1}{2} > 0 \Longrightarrow 2\bar{x}_1 + \bar{x}_2 = 2, \ \bar{x}_1 = 1 > 0 \Longrightarrow 2y_1 - y_2 = 1$$

Equivalent formulation

Back to a zero sum game described by a payoff matrix P. We can assume, w.l.o.g., that $p_{ij} > 0$ for all i, j. This implies $v > 0$

Set $\alpha_i = \frac{x_i}{v}$. Then $\sum x_i = 1$ becomes $\sum \alpha_i = \frac{1}{v}$ and maximizing v is equivalent to minimizing $\sum \alpha_i$. Set $\beta_j = \frac{y_j}{v}$ and do the same as before.

Consider the two problems in duality

$$(P) \quad \begin{cases} \min c^t \alpha \\ A\alpha \geq b \\ \alpha \geq 0 \end{cases} \qquad (D) \quad \begin{cases} \max b^t \beta \\ A^t \beta \leq c \\ \beta \geq 0 \end{cases}$$

where $c^t = (1, \dots, 1)$, $b^t = (1, \dots, 1)$, $A = P^t$.
Denote by v the common value of the two problems. We have

- x is optimal strategy for Pl1 if and only if $x = v\alpha$ for some α optimal solution of (P)
- y is optimal strategy for Pl1 if and only if $y = v\beta$ for some β optimal solution of (D)

Complementarity conditions in zero sum games

Write again the complementarity conditions for the above problems, being x,y strategies for the two players:

$$(CC)\begin{cases} (\forall i = 1, \dots, n) & \bar{x}_i > 0 \Rightarrow \sum_{j=1}^{m} p_{ij}\bar{y}_j = v \\ (\forall j = 1, \dots, m) & \bar{y}_j > 0 \Rightarrow \sum_{i=1}^{n} p_{ji}\bar{x}_i = v \end{cases}$$

Interpretation:
Since $\bar{y}$ is optimal for Pl2, it is $\sum_{j=1}^{m} p_{ij}\bar{y}_j = v$ for all i, and thus $x_i > 0$ implies that the row i is optimal for Pl1. And conversely for Pl2.

Summarizing

- A finite zero sum game has always rational outcome in mixed strategies
- The set of optimal strategies for the players is a nonempty closed convex set
- The outcome, at each pair of optimal strategies, is the common conservative value v of the players

Symmetric games

Definition

A square matrix $n \times n$ $P = (p_{ij})$ is said to be antisymmetric provided $p_{ij} = -p_{ji}$ for all $i, j = 1, \ldots, n$.
A (finite) zero sum game is said to be fair if the associated matrix is antisymmetric

In fair games there is no favorite player

Fair outcome

Proposition

If $P = (p_{ij})$ is antisymmetric the value is 0 and $\bar{x}$ is an optimal strategy for Pl1 if and only if it is optimal for Pl2

Proof Since

$$x^t P x = (x^t P x)^t = x^t P^t x = -x^t P x$$

$f(x,x) = 0$ for all x: this implies $v_1 \leq 0, v_2 \geq 0$

Then $v = 0$

If $\bar{x}$ is optimal for the first player, $\bar{x}^t P y \geq 0$ for all y and transposing

$y^t P \bar{x} \leq 0$ for all $y \in \Sigma_n$,

thus $\bar{x}$ is optimal also for the second player, and conversely ■

Finding optimal strategies in a fair game

Need to solve the system of inequalities

$$
\begin{aligned}
& x_1 p_{11} + \cdots + x_n p_{n1} \geq 0 \\
& \ldots \\
& x_1 p_{1j} + \cdots + x_n p_{nj} \geq 0 \\
& \ldots \\
& x_1 p_{1n} + \cdots + x_n p_{nn} \geq 0
\end{aligned} \tag{3}
$$

with the extra conditions:

$$x_i \geq 0, \qquad \sum_{i=1}^{n} x_i = 1$$

A proposed exercise

Example

Find the optimal strategies of the following fair game:

$$P = \begin{pmatrix} 0 & 3 & -2 & 0 \\ -3 & 0 & 0 & 4 \\ 2 & 0 & 0 & -3 \\ 0 & -4 & 3 & 0 \end{pmatrix}$$