Zero sum games

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General form

Definition

A two player *zero sum game* *in strategic form is the triplet* $(X, Y, f : X \times Y \to \mathbb{R})$

$f(x, y)$ is what Pl1 gets from Pl2, when they play x, y respectively. Thus $g = -f$.

Finite game

In the finite case $X = \{1, 2, \dots, n\}$, $Y = \{1, 2, \dots, m\}$ the game is described by a payoff matrix P

Example

$$P = \begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}$$

Pl1 selects row i, Pl2 selects column j.

A different approach to solve them

$$\begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}.$$

Pl1 can guarantee herself to get at least

$$v_1 = \max_i \min_j p_{ij}$$

Pl2 can guarantee himself to pay no more than

$$v_2 = \min_j \max_i p_{ij}$$

$\min_j p_{1j} = 1$, $\min_j p_{2j} = 5$, $\min_j p_{3j} = 0$ $\quad v_1 = 5$
$\min_i p_{i1} = 8$, $\min_j p_{i2} = 5$, $\min_j p_{i3} = 8$, $\quad v_2 = 5$

Rational outcome 5.Rational behavior ($\bar{\imath} = 2, \bar{\jmath} = 2$).

Alternative idea of solution

Suppose $v_1 = v_2 := v$, denote by $\bar{\imath}$ ($\bar{\jmath}$) the row (column) such that $p_{\bar{\imath}j} \geq v$ for all j ($p_{i\bar{\jmath}} \leq v$ for all i).

Then $p_{\bar{\imath}\bar{\jmath}} = v$ and $p_{\bar{\imath}\bar{\jmath}} = v$ is the rational outcome of the game

Remark

$\bar{\imath}$ ($\bar{\jmath}$) is an optimal strategy for Pl1 (for Pl2), because he cannot get more (cannot pay less) than v (since v is the conservative value of the second (first) player)

For arbitrary games

$(X, Y, f : X \times Y \to \mathbb{R})$

The players can guarantee to themselves (almost):

Pl1: $v_1 = \sup_x \inf_y f(x, y)$

PL2: $v_2 = \inf_y \sup_x f(x, y)$

v_1, v_2 are the conservative values of the players

Optimality

Suppose $v_1 = v_2 := v$, strategies $\bar{x}$ and $\bar{y}$ exist such that

$$f(\bar{x}, y) \geq v, \quad f(x, \bar{y}) \leq v$$

for all y and for all x

Then $f(\bar{x}, \bar{y}) = v$ is the rational outcome of the game

$\bar{x}$ is an optimal strategy for Pl1, $\bar{y}$ is an optimal strategy for Pl2

$v_1 \leq v_2$

Proposition

Let X, Y be any sets and let $f : X \times Y \to \mathbb{R}$ be an arbitrary function. Then

$$\sup_x \inf_y f(x,y) \leq \inf_y \sup_x f(x,y)$$

Proof Observe that, for all x, y,

$$\inf_y f(x,y) \leq f(x,y) \leq \sup_x f(x,y)$$

Thus

$$\inf_y f(x,y) \leq \sup_x f(x,y)$$

Since the left hand side of the above inequality does not depend on y and the right hand side on x, the thesis follows ■

In every game $v_1 \leq v_2$, as expected

Equality need not hold

Example

$$P = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

$v_1 = -1$, $v_2 = 1$

Nothing unexpected...

Case $v_1 < v_2$

Finite case: mixed strategies. Game: $n \times m$ matrix P.

Strategy space for Pl1:

$$\Sigma_n = \{x = (x_1, \dots, x_n) : x_i \geq 0, \sum_{i=1}^{n} x_i = 1\}$$

Strategy space for Pl2:

$$\Sigma_m = \{y = (y_1, \dots, y_m) : y_j \geq 0, \sum_{j=1}^{m} y_j = 1\}$$

$$f(x, y) = \sum_{i=1,\dots,n, j=1,\dots,m} x_i y_j p_{ij} = x^t P y$$

The mixed extension of the initial game P: $(\Sigma_n, \Sigma_m, f(x, y) = x^t P y)$

To prove existence of a rational outcome

What must be proved, to have existence of a rational outcome:

1) $v_1 = v_2$
2) there exists $\bar{x}$ fulfilling

$$v_1 = \sup_x \inf_y f(x, y) = \inf_y f(\bar{x}, y)$$

3) there exists $\bar{y}$ fulfilling

$$v_2 = \inf_y \sup_x f(x, y) = \sup_x f(x, \bar{y})$$

In the finite case $\bar{x}$ and $\bar{y}$ fulfilling 1) and 2) always exist; thus existence is equivalent to 1)

The von Neumann theorem

Theorem

A two player, finite, zero sum game as described by a payoff matrix P has a rational outcome

Convexity (1)

Definition

A set $C \subset \mathbb{R}^n$ is said to be convex provided $x, y \in C$, $\lambda \in [0, 1]$ imply:

$$\lambda x + (1 - \lambda) y \in C$$

Remark

- *The intersection of an arbitrary family of convex sets is convex*
- *A closed convex set with nonempty interior coincides with the closure of its internal points*

Definition

We shall call a convex combination of elements $x_1, \dots, x_n$ any vector x of the form

$$x = \lambda_1 x_1 + \cdots + \lambda_n x_n,$$

with $\lambda_1 \geq 0, \dots, \lambda_n \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$

Convexity (2)

Proposition

A set C is convex if and only if for every $\lambda_1 \geq 0, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$, for every $c_1, \ldots, c_n \in C$, for all n, then $\sum_{i=1}^n \lambda_i c_i \in C$

If C is not convex, then there is a smallest convex set containing C: it is the intersection of all convex sets containing C

Definition

The convex hull of a set C, denoted by co C, is:

$$\mathrm{co}\, C \stackrel{\text{def}}{=} \bigcap_{A \in \mathcal{C}} A$$

where $\mathcal{C} = \{A : C \subset A \ \wedge \ A \text{ is convex}\}$

Convexity (3)

Proposition

Given a set C, then

$$\operatorname{co} C = \{\sum_{i=1}^{n} \lambda_i c_i : \ \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, c_i \in C \ \forall i, n \in \mathbb{N}\}$$

Theorem

Given a closed convex set C and a point x outside C, there is a unique element $p \in C$ such that

$$||p - x|| \leq ||c - x||$$

for all $c \in C$. p is characterized by

- $p \in C$
- $\langle x - p, c - p \rangle \leq 0$ *for all* $c \in C$

A first separation result

Theorem

Let C be a convex proper subset of the Euclidean space $\mathbb{R}^l$, let $\bar{x} \in$ cl C^c. Then there is an element $0 \neq x^* \in \mathbb{R}^l$ such that:

$$\langle x^*, c\rangle \geq \langle x^*, \bar{x}\rangle$$

$\forall c \in C$

Proof Suppose $\bar{x} \notin$ cl C and call p its projection on cl C. Then $\langle \bar{x} - p, c - p\rangle \leq 0$ for all $c \in C$. Setting $x^* = p - \bar{x}$

$$\langle x^*, c - \bar{x}\rangle \geq \|x^*\|^2$$

implying

$$\langle x^*, c\rangle \geq \langle x^*, \bar{x}\rangle$$

$\forall c \in C$. We can choose $\|x^*\| = 1$. If $\bar{x} \in \overline{C} \setminus C$, take a sequence $\{x_n\} \subset C^c$ such that $x_n \to \bar{x}$. From the first step of the proof, find norm one x_n^* such that

$$\langle x_n^*, c\rangle \geq \langle x_n^*, x_n\rangle$$

$\forall c \in C$. Thus, possibly passing to a subsequence, we can suppose $x_n^* \to x^*$, where $\|x^*\| = 1$ (so that $x^* \neq 0$). Now take the limit in the above inequality, to get:

$$\langle x^*, c\rangle \geq \langle x^*, \bar{x}\rangle$$

$\forall c \in C$ ■

Separating hyperplane

Corollary

Let C be a closed convex set in a Euclidean space, let x be on the boundary of C. Then there is a hyperplane containing x and leaving all of C in one of the halfspaces determined by the hyperplane

The hyperplane whose existence is established in the Corollary is said the be an hyperplane supporting C at x

Corollary

Let C be a closed convex set in a Euclidean space. Then C is the intersection of all halfspaces containing it

The separation result

Theorem

Let A, C be closed convex subsets of $\mathbb{R}^l$ such that $\operatorname{int} A$ is nonempty and $\operatorname{int} A \cap C = \emptyset$. Then there is $0 \neq x^*$ such that

$$\langle x^*, a\rangle \geq \langle x^*, c\rangle$$

$\forall a \in A$, $\forall c \in C$

Proof Since $0 \in (\text{ int } A - C)^c$, we can apply the previous separation theorem to find $x^* \neq 0$ such that

$$\langle x^*, x\rangle \geq 0$$

$\forall x \in \text{int } A - C$. Thus:

$$\langle x^*, a\rangle \geq \langle x^*, c\rangle$$

$\forall a \in \text{int } A$, $\forall c \in C$. This implies

$$\langle x^*, a\rangle \geq \langle x^*, c\rangle$$

$\forall a \in \text{cl int } A = A$, $\forall c \in C$ ∎

The proof of vN theorem

Proof Suppose all entries p_{ij} of the matrix P are positive. Consider the vectors $p_1, \dots, p_m$ of $\mathbb{R}^n$, where p_j denotes the j^{th} column of the matrix P. Call C the convex hull of these vectors, set

$$Q_t \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n : x_i \le t\} \ \wedge \ v = \sup\{t \ge 0 : Q_t \cap C = \emptyset\}$$

Q_v and C can be (weakly) separated by an hyperplane: there are coefficients $\bar{x}_1, \dots, \bar{x}_n$, not all zero, and $b \in \mathbb{R}$ such that

$$\sum_{i=1}^{n} \bar{x}_i u_i = \langle \bar{x}, u \rangle \le b \le \sum_{i=1}^{n} \bar{x}_i w_i = \langle \bar{x}, w \rangle$$

for all $u = (u_1, \dots, u_n) \in Q_v$, $w = (w_1, \dots, w_n) \in C$. It holds

1. All $\bar{x}_i$ must be nonnegative and, since they cannot be all zero, we can assume $\sum \bar{x}_i = 1$
2. $b = v$; First of all, since $\bar{v} := (v, \dots, v) \in Q_v$ we have, from $\langle \bar{x}, \bar{v} \rangle \le v$ that $b \ge v$. Suppose now $b > v$, and take $a > 0$ so small that $b > v + a$. Then $\sup\{\sum_{i=1}^{n} \bar{x}_i u_i \ : \ u \in Q_{v+a}\} < b$, and this implies $Q_{v+a} \cap C = \emptyset$, against the definition of v
3. $Q_v \cap C \ne \emptyset$

Given $\beta \in \Sigma_m$, let $w = \sum_{j=1}^{m} \beta_j p_j \in C$ (thus $w_i = \sum_{j=1}^{m} \beta_j p_{ij}$). Thus

$$f(\bar{x}, \beta) = \sum_{i,j} \bar{x}_i \beta_j p_{ij} = \sum_{i=1}^{n} \bar{x}_i w_i \ge v$$

Now, let $\bar{w} \in Q_v \cap C$. Since $\bar{w} \in C$, then $\bar{w} = \sum_{j=1}^{m} \bar{\beta}_j p_j$, for some $\Sigma_m \ni \bar{\beta} = (\bar{\beta}_1, \dots, \bar{\beta}_m)$. Since $\bar{w} \in Q_v$, then $\bar{w}_i \le v$ for all i. Thus, for all $\lambda \in \Sigma_n$, we get

$$f(\lambda, \bar{\beta}) = \sum_{ij} \lambda_i \bar{\beta}_j p_{ij} = \sum_i \lambda_i \bar{w}_i \le v \sum_i \lambda_i = v \quad \blacksquare$$

Finding optimal strategies:Pl1

Pl1 must choose a probability distribution $\Sigma_n \ni x = (x_1, \dots, x_n)$:

$$\begin{aligned} & x_1 p_{11} + \cdots + x_n p_{n1} \geq v \\ & \dots \\ & x_1 p_{1j} + \cdots + x_n p_{nj} \geq v \\ & \dots \\ & x_1 p_{1m} + \cdots + x_n p_{nm} \geq v \end{aligned}$$

where v must be as large as possible

Finding optimal strategies:Pl2

Pl2 must choose a probability distribution $\Sigma_m \ni y = (y_1, \dots, y_m)$:

$$
\begin{aligned}
& y_1 p_{11} + \cdots + y_m p_{1m} \leq w \\
& \dots \\
& y_1 p_{i1} + \cdots + y_m p_{im} \leq w \\
& \dots \\
& y_1 p_{n1} + \cdots + y_m p_{nm} \leq w
\end{aligned}
$$

where w must be as small as possible

In matrix form

Pl1:

$$\begin{cases} \max_{x,v} v : \\ P^t x \geq v 1_m \\ x \geq 0 \quad \langle 1, x \rangle = 1 \end{cases} \tag{1}$$

Pl2:

$$\begin{cases} \min_{y,w} w : \\ P y \leq w 1_n \\ y \geq 0 \quad \langle 1, y \rangle = 1 \end{cases} \tag{2}$$

Easy to see that (1) and (2) are dual problems, they are feasible, and the two values agree

Summarizing

A finite zero sum game has always rational outcome in mixed strategies

The set of optimal strategies for the players is a nonempty closed convex set

The outcome, at each pair of optimal strategies, is the common conservative value v of the players

Symmetric games

Definition

A square matrix $n \times n$ $P = (p_{ij})$ *is said to be* ***antisymmetric*** *provided* $p_{ij} = -p_{ji}$ *for all* $i, j = 1, \ldots, n$.
A (finite) zero sum game is said to be ***fair*** *if the associated matrix is antisymmetric*

In fair games there is no favorite player

Fair outcome

Proposition

If $P = (p_{ij})$ is antisymmetric the value is 0 and $\bar{x}$ is an optimal strategy for Pl1 if and only if it is optimal for Pl2

Proof Since

$$x^t P x = (x^t P x)^t = x^t P^t x = -x^t P x$$

$f(x, x) = 0$ for all x thus $v_1 \leq 0, v_2 \geq 0$

Then $v = 0$

If $\bar{x}$ is optimal for the first player, $\bar{x}^t P y \geq 0$ for all y

Thus $y^t P \bar{x} \leq 0$ for all $y \in \Sigma_n$, and

$\bar{x}$ is optimal for the second player ■

Finding optimal strategies in a fair game

Need to solve the system of inequalities

$$
\begin{aligned}
&x_1 p_{11} + \cdots + x_n p_{n1} \geq 0 \\
&\ldots \\
&x_1 p_{1j} + \cdots + x_n p_{nj} \geq 0 \\
&\ldots \\
&x_1 p_{1m} + \cdots + x_n p_{nm} \geq 0
\end{aligned}
\tag{3}
$$

with the extra conditions:

$$
x_i \geq 0, \qquad \sum_{i=1}^{n} x_i = 1
$$

A proposed exercise

Example

Find the optimal strategies of the following fair game:

$$P = \begin{pmatrix} 0 & 3 & -2 & 0 \\ -3 & 0 & 0 & 4 \\ 2 & 0 & 0 & -3 \\ 0 & -4 & 3 & 0 \end{pmatrix}$$

Toward Indifference Principle

In the system (with v unknown!)

$$
\begin{aligned}
&x_1 p_{11} + \cdots + x_n p_{n1} \geq v \\
&\ldots \\
&x_1 p_{1j} + \cdots + x_n p_{nj} \geq v \\
&\ldots \\
&x_1 p_{1m} + \cdots + x_n p_{nm} \geq v
\end{aligned}
\tag{4}
$$

when a strict inequality is possible?

Suppose $\bar{x}$ is optimal for Pl1 and

$\bar{x}_1 p_{1j} + \cdots + \bar{x}_n p_{nj} > v$

Then Pl2 never plays column j

Otherwise Pl1 would get more than v playing $\bar{x}$

The Principle

There is a nonempty set of indices $J_1 = \{j_1, \dots, j_k\}$ such that

$$x_1 p_{1j_1} + \dots + x_n p_{nj_1} = x_1 p_{1j_2} + \dots + x_n p_{nj_2} = \dots = x_1 p_{1j_k} + \dots + x_n p_{nj_k}$$

and

$$x_1 p_{1j_1} + \dots + x_n p_{nj_1} > x_1 p_{1j} + \dots + x_n p_{nj}$$

for all $j \notin J_1$

J_1 is the set of columns played with positive probability by Pl2 at some optimal strategy

Also true: if $j \notin J_1$ there exists an optimal strategy for Pl1 providing her a payoff $> v$ against column j