Zero sum games

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Summary of the slides

1. Zero sum game in strategic form
2. Conservative values of the players
3. Optimal strategies for the players and common value
4. $v_1 \leq v_2$ for arbitrary games
5. Mixed extension of the (finite) zero sum game
6. The von Neumann theorem
7. Finding optimal strategies for the players as LP problems
8. Some basics of Linear Programming
9. Duality: the weak and the strong duality theorems
10. Complementarity conditions
11. Equivalent formulations for finding optimal strategies
12. Complementarity conditions in the zero sum games
13. Nash equilibria profiles, optimal strategies and the value in zero sum game
14. Fair games

General form

Definition

A two player zero sum game in strategic form is the triplet $(X, Y, f : X \times Y \to \mathbb{R})$

X is the strategy space of Pl1, Y the strategy space of Pl2, $f(x, y)$ is what Pl1 gets from Pl2, when they play x, y respectively. Thus f is the utility function of Pl1, while for Pl2 the utility function g is $g = -f$.

Finite game

In the finite case $X = \{1, 2, \ldots, n\}$, $Y = \{1, 2, \ldots, m\}$ the game is described by a payoff matrix P

Example

$$P = \begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}$$

Pl1 selects row i, Pl2 selects column j.
In general

$$\begin{pmatrix} p_{11} & \cdots & p_{1m} \\ \cdots & \cdots & \cdots \\ p_{n1} & \cdots & p_{nm} \end{pmatrix}$$

where p_{ij} is the payment of Pl2 to Pl1 when they play i, j respectively.

How to solve them

Consider the game

$$\begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}$$

- $\min_j p_{1j} = 1$, $\min_j p_{2j} = 5$, $\min_j p_{3j} = 0 \quad v_1 = 5$
- $\max_i p_{i1} = 8$, $\max_i p_{i2} = 5$, $\max_i p_{i3} = 8, \quad v_2 = 5$

Thus

- Pl1 can guarantee herself to get at least

$$v_1 = \max_i \min_j p_{ij}$$

- Pl2 can guarantee himself to pay no more than

$$v_2 = \min_j \max_i p_{ij}$$

In the example $v_1 = v_2 = 5$ and
Rational outcome 5.Rational behavior ($\bar{\imath} = 2, \bar{\jmath} = 2$)

Alternative idea of solution

Suppose

- $v_1 = v_2 := v$,
- $\bar{\imath}$ a row such that $p_{\bar{\imath}j} \geq v$ for all j
- $(\bar{\jmath})$ a column such that $p_{i\bar{\jmath}} \leq v$ for all i

Then $p_{\bar{\imath}\bar{\jmath}} = v$ and $p_{\bar{\imath}\bar{\jmath}} = v$ is the rational outcome of the game

Remark

- $\bar{\imath}$ *is an optimal strategy for Pl1, because she cannot get more than* v*, since* $v = v_2$ *is the conservative value of the second player*
- $\bar{\jmath}$ *is an optimal strategy for Pl2, because he cannot pay less than* v*, since* $v = v_1$ *is the conservative value of the first player*

Remark

Observe $\bar{\imath}$ *maximizes the function* $\alpha(i) = \min_j p_{ij}$, $\bar{\jmath}$ *minimizes the function* $\beta(j) = \max_i p_{ij}$

For arbitrary games

$(X, Y, f : X \times Y \to \mathbb{R})$

The players can guarantee to themselves (almost):

Pl1: $v_1 = \sup_x \inf_y f(x, y)$

PL2: $v_2 = \inf_y \sup_x f(x, y)$

v_1, v_2 are the conservative values of the players

If $v_1 = v_2$, we set $v = v_1 = v_2$ and we say that the game has value v

Optimality

Suppose

1. $v_1 = v_2 := v$
2. there exists strategy $\bar{x}$ such that $f(\bar{x}, y) \geq v$ for all $y \in Y$
3. there exists strategy $\bar{y}$ such that $f(x, \bar{y}) \leq v$ for all $x \in X$

Then

- v is the rational outcome of the game
- $\bar{x}$ is an optimal strategy for Pl1
- $\bar{y}$ is an optimal strategy for Pl2

Observe

- $\bar{x}$ is optimal for Pl1 since it maximizes the function $\alpha(x) = \inf_y f(x, y)$
- $\bar{x}$ is optimal for Pl2 since it minimizes the function $\beta(y) = \sup_x f(x, y)$

$\alpha(x)$ is the value of the optimal choice of Pl2 if he knows that Pl1 plays x; symmetrically for $\beta(y)$

$v_1 \leq v_2$

Proposition

Let X, Y be nonempty sets and let $f : X \times Y \to \mathbb{R}$ be an arbitrary real valued function. Then

$$\sup_x \inf_y f(x, y) \leq \inf_y \sup_x f(x, y)$$

Proof Observe that, for all x, y,

$$\inf_y f(x, y) \leq f(x, y) \leq \sup_x f(x, y)$$

Thus

$$\alpha(x) = \inf_y f(x, y) \leq \sup_x f(x, y) = \beta(y)$$

Since for all $x \in X$ and $y \in Y$ it holds

$$\alpha(x) \leq \beta(y)$$

it follows

$$v_1 = \sup_x \alpha(x) \leq \inf_y \beta(y) = v_2 \quad \blacksquare$$

As a consequence, in every game $v_1 \leq v_2$

Equality need not hold

Example

$$P = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

$v_1 = -1$, $v_2 = 1$

Nothing unexpected...

Case $v_1 < v_2$

Finite case: mixed strategies. Game: $n \times m$ matrix P.

Strategy space for Pl1:

$$\Sigma_n = \{x = (x_1, \dots, x_n) : x_i \geq 0, \sum_{i=1}^{n} x_i = 1\}$$

Strategy space for Pl2:

$$\Sigma_m = \{y = (y_1, \dots, y_m) : y_j \geq 0, \sum_{j=1}^{m} y_j = 1\}$$

$$f(x, y) = \sum_{i=1,\dots,n, j=1,\dots,m} x_i y_j p_{ij} = x^t P y$$

The mixed extension of the initial game P: $(\Sigma_n, \Sigma_m, f(x, y) = x^t P y)$

Remark

Σ_n is called the fundamental simplex in $\mathbb{R}^n$: it is the smallest convex set containing the extreme points $(1, 0, \dots, 0), \dots, (0, \dots, 1)$. The extreme points of the simplex correspond to pure strategies.

The von Neumann theorem

Theorem

A two player, finite, zero sum game as described by a payoff matrix P has a rational outcome

Finding optimal strategies

In a finite zero sum game P to find optimal strategies for the players, we need to find $\bar{x}, \bar{y}$ fulfilling:

- $\bar{x}^t P y \geq v \quad \forall y$
- $x^t P \bar{y} \leq v \quad \forall x.$

Two facts must to be taken into account

- the value v is unknown
- an optimal strategy for a player, fixed the strategy of the other player, can be found among the pure strategies.

Taking the above remarks into account, the problem of finding optimal strategies can be reformulated as a Linear Programming problem.

Finding optimal strategies:Pl1

Pl1 must choose a probability distribution $\Sigma_n \ni x = (x_1, \dots, x_n)$ in order to maximize z with the constraints:

$$\begin{aligned}
& x_1 p_{11} + \dots + x_n p_{n1} \geq z \\
& \dots \\
& x_1 p_{1j} + \dots + x_n p_{nj} \geq z \\
& \dots \\
& x_1 p_{1m} + \dots + x_n p_{nm} \geq z
\end{aligned}$$

If $x \in \Sigma_n$ satisfies the above system of inequalities, then Pl1 obtains at least z against all columns of the Payoff matrix, i.e. against all pure strategies, corresponding to the extreme points of the simplex. Such x then guarantees Pl1 to get at least v also against every mixed strategy of Pl2. Thus a solution is an optimal strategy of Pl1 and the value of the problem is exactly v, the value of the game.

Finding optimal strategies:Pl2

Following the same argument: Pl2 must choose a probability distribution $\Sigma_m \ni y = (y_1, \dots, y_m)$ in order to minimize w with the constraints:

$$
\begin{aligned}
& y_1 p_{11} + \cdots + y_m p_{1m} \leq w \\
& \dots \\
& y_1 p_{i1} + \cdots + y_m p_{im} \leq w \\
& \dots \\
& y_1 p_{n1} + \cdots + y_m p_{nm} \leq w
\end{aligned}
$$

Again the value of this problem is v, the value of the game, thanx to von Neumann theorem.

In matrix form

Pl1:

$$\begin{cases} \max_{x,v} v : \\ P^t x \geq v 1_m \\ x \geq 0 \quad 1^t x = 1 \end{cases} \tag{1}$$

Pl2:

$$\begin{cases} \min_{y,w} w : \\ Py \leq w 1_n \\ y \geq 0 \quad 1^t y = 1 \end{cases} \tag{2}$$

where 1 is a vector of the right dimension made by all 1's.
(1) and (2) are dual Linear Programming (LP) problems.

Dual linear programs: Form 1

Definition

The following two linear programs are said to be in duality

$$(P) \quad \begin{cases} \min c^t x \\ Ax \geq b \\ x \geq 0 \end{cases} \qquad\qquad (D) \quad \begin{cases} \max b^t y \\ A^T y \leq c \\ y \geq 0 \end{cases}$$

The min problem is called primal problem and the max is called dual problem.

Dual linear programs: Form 2

Definition

The following two linear programs are said to be in duality

$$(P)\quad \begin{cases} \min c^t x \\ Ax \geq b \end{cases} \qquad\qquad (D)\quad \begin{cases} \max b^t y \\ A^T y = c \\ y \geq 0 \end{cases}$$

There is a standard way to pass from the first form of (P) (where non negativity constraint are present) to the second form of (P), and conversely; dualizing the problems leads to equivalent form of the dual problems.

Feasibility of dual programs

Problem is said feasible if there is at least one vector fulfilling the constraints.
Easy examples show that, given two problems in duality,

- They can be both infeasible
- Only one can be feasible
- Both can be feasible

Example 1

Consider

$$\begin{cases} \min x_1 + x_2 \\ x_1 + 2x_2 \geq 1 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Its dual is

$$\begin{cases} \max y \\ y \leq 1 \\ 2y \leq 1 \\ y \geq 0 \end{cases}$$

Since $(x_1, x_2) = (0, \frac{1}{2})$ fulfills the constraints of the primal problem and $y = \frac{1}{2}$ fulfills the constraints of the dual problem, they are both feasible.

Examples 2,3

Consider

$$\begin{cases} \min x_1 - x_2 \\ x_1 + x_2 \geq 2 \\ -x_1 - x_2 \geq -1 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Its dual is

$$\begin{cases} \max 2y_1 - y_2 \\ y_1 - y_2 \leq 1 \\ y_1 - y_2 \leq -1 \\ y_1 \geq 0, y_2 \geq 0 \end{cases}$$

The primal is infeasible (no $x = (x_1, x_2)$ can fulfill at the same time the inequalities $x_1 + x_2 \geq 2$ and $-x_1 - x_2 \geq -1$), while for all $n \geq 0$ $(n, n+1)$ is feasible in the dual.

Taking $A = 0$, $b = (1, \dots, 1)$ and $c = (-1, \dots, -1)$ shows that both problems can be infeasible.

Weak duality theorem

Theorem

Let v be the value of the primal min problem and V the value of the dual max problem. Then

$$v \geq V$$

Proof

Form 1:

$$c^t x \geq (A^t y)^t x = y^t A x \geq y^t b = b^t y$$

Since this is true for all admissible x and y the result follows.

Form 2:

$$c^t x = (A^t y)^t x = y^t A x \geq y^t b = b^t y$$

■

Strong duality theorem

Theorem

- *If the primal and dual problems are feasible, then both problems have optimal solutions* $\bar{x}, \bar{y}$ *and the optimal values coincide*

$$v = c^t \bar{x} = b^t \bar{y} = V.$$

  In this case we say that there is no duality gap.
- *If the primal is feasible and the dual is infeasible, then* $v = V = -\infty$
- *If the primal is infeasible and the dual is feasible, then* $v = V = +\infty$
- *If both the primal and the dual are infeasible, then* $v = \infty > V = -\infty$

Corollary

If one problem is feasible and has an optimal solution, then also the dual problem is feasible and has solutions. Moreover there is no duality gap.

Complementarity conditions: Form 1

$$(P) \begin{cases} \min c^t x \\ Ax \geq b, x \geq 0 \end{cases} \quad ; \quad (D) \begin{cases} \max b^t y \\ A^T y \leq c, y \geq 0 \end{cases}$$

Theorem

Let $\bar{x}, \bar{y}$ be primal and dual feasible. Then $\bar{x}, \bar{y}$ are simultaneously optimal iff

$$(CC) \begin{cases} (\forall i = 1, \dots, n) & \bar{x}_i > 0 \Rightarrow \sum_{j=1}^m a_{ji}\bar{y}_j = c_i \\ (\forall j = 1, \dots, m) & \bar{y}_j > 0 \Rightarrow \sum_{i=1}^n a_{ij}\bar{x}_i = b_j \end{cases}$$

Proof Since $c^t x \geq y^t Ax \geq b^t y$ it follows that $\bar{x}, \bar{y}$ are optimal iff

$$c^t \bar{x}^t = \bar{y}^t A\bar{x} = b^t \bar{y}$$

This is equivalent to

$$\bar{x}^t(A^t\bar{y} - c) = 0 \qquad \text{and} \qquad \bar{y}^t(A\bar{x} - b) = 0$$

Since $\bar{x}, \bar{y} \geq 0$ and $A\bar{x} \geq b, A^t\bar{y} \leq c$ the latter are equivalent to (CC). ■

An example

Consider

$$\begin{cases} \min x_1 + x_2 : \\ 2x_1 + x_2 \geq 2 \\ x_1 + 2x_2 \leq 2 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Its dual is

$$\begin{cases} \max 2y_1 - 2y_2 : \\ 2y_1 - y_2 \leq 1 \\ y_1 - 2y_2 \leq 1 \\ y_1 \geq 0, y_2 \geq 0 \end{cases}$$

We have $v = 1$, $(\bar{x}_1, \bar{x}_2) = (1, 0)$; $V = 1$, $(\bar{y}_1, \bar{y}_2) = (\frac{1}{2}, 0)$.

Check of the complementarity conditions:

$$\bar{y}_1 = \frac{1}{2} > 0 \Longrightarrow 2\bar{x}_1 + \bar{x}_2 = 2,\ \bar{x}_1 = 1 > 0 \Longrightarrow 2y_1 - y_2 = 1$$

Equivalent formulation

Back to a zero sum game described by a payoff matrix P. We can assume, w.l.o.g., that $p_{ij} > 0$ for all i, j. This implies $v > 0$

Set $\alpha_i = \frac{x_i}{v}$. Then $\sum x_i = 1$ becomes $\sum \alpha_i = \frac{1}{v}$ and maximizing v is equivalent to minimizing $\sum \alpha_i$. Set $\beta_j = \frac{y_j}{v}$ and do the same as before.

Consider the two problems in duality

$$(P) \quad \begin{cases} \min c^t \alpha \\ A\alpha \geq b \\ \alpha \geq 0 \end{cases} \qquad (D) \quad \begin{cases} \max b^t \beta \\ A^t \beta \leq c \\ \beta \geq 0 \end{cases}$$

where $c^t = (1, \dots, 1)$, $b^t = (1, \dots, 1)$, $A = P^t$.
Denote by v the common value of the two problems. We have

- x is optimal strategy for Pl1 if and only if $x = v\alpha$ for some α optimal solution of (P)
- y is optimal strategy for Pl2 if and only if $y = v\beta$ for some β optimal solution of (D)

Complementarity conditions in zero sum games

Write again the complementarity conditions for the above problems, being x,y strategies for the two players:

$$(CC)\begin{cases} (\forall i=1,\dots,n) & \bar{x}_i>0 \Rightarrow \sum_{j=1}^{m} p_{ij}\bar{y}_j = v \\ (\forall j=1,\dots,m) & \bar{y}_j>0 \Rightarrow \sum_{i=1}^{n} p_{ji}\bar{x}_i = v \end{cases}$$

Interpretation:

1. Since $\bar{y}$ is optimal for Pl2, he is able to pay no more than v against all strategies of the first player
2. If $\bar{x}_i > 0$ then Pl1 plays row i with positive probability

Thus the complementarity conditions show (one more time!) that, if played with positive probability, the row i must be optimal for Pl1 (since by (1) we know that she gets less or equal to v by playing the other rows).

Summarizing

- A finite zero sum game has always rational outcome in mixed strategies
- The set of optimal strategies for the players is a nonempty closed convex set
- The outcome, at each pair of optimal strategies, is the common conservative value v of the players

The Nash equilibria of a zero sum game

Theorem

Let X, Y be (nonempty) sets and $f : X \times Y \to \mathbb{R}$ a function. Then the following are equivalent:

1. *The pair $(\bar{x}, \bar{y})$ fulfills*

$$f(x, \bar{y}) \leq f(\bar{x}, \bar{y}) \leq f(\bar{x}, y) \quad \forall x \in X, \ \forall y \in Y$$

2. *The following conditions are satisfied:*
(i) $\inf_y \sup_x f(x, y) = \sup_x \inf_y f(x, y)$ *(The conservative values agree)*
(ii) $\inf_y f(\bar{x}, y) = \sup_x \inf_y f(x, y)$ *($\bar{x}$ is optimal for Pl 1)*
(iii) $\sup_x f(x, \bar{y}) = \inf_y \sup_x f(x, y)$ *($\bar{y}$ is optimal for Pl 2)*

Proof

Proof 1) implies 2). From 1) we get:

$$\inf_y \sup_x f(x,y) \leq \sup_x f(x,\bar{y}) = f(\bar{x},\bar{y}) = \inf_y f(\bar{x},y) \leq \sup_x \inf_y f(x,y)$$

Since $v_1 \leq v_2$ always holds, all above inequalities are equalities

Conversely, suppose 2) holds Then

$$\inf_y \sup_x f(x,y) \stackrel{(iii)}{=} \sup_x f(x,\bar{y}) \geq f(\bar{x},\bar{y}) \geq \inf_y f(\bar{x},y) \stackrel{(ii)}{=} \sup_x \inf_y f(x,y)$$

Because of (i), all inequalities are equalities and the proof is complete ■

Finite games

The above theorem, when applied to finite games and mixed strategies, provides the next

Corollary

The following are equivalent:

- $(\bar{x}, \bar{y})$ *is a NE with* $\bar{x}^t P \bar{y} = v$
- *the game has value* v, $\bar{x}$ *is optimal for Pl1,* $\bar{y}$ *is optimal for Pl2*

As a consequence of the theorem

Any $(\bar{x}, \bar{y})$ Nash equilibrium of the zero sum game provides optimal strategies for the players

Any pair of optimal strategies for the players provides a Nash equilibrium for the zero sum game

Thus Nash theorem generalizes von Neumann's

A comment

Remark

We defined two rationality paradigms for zero sum games

- *common conservative value of the game, and optimal strategies for the players*
- *Nash equilibria*

From the above result, they are the same rationality paradigm. Moreover, Von Neumann approach with conservatives values shows that, in the particular case of the zero sum game:

- *Players can find their optimal behavior independently for the other players*
- *Any pair of optimal strategies provides a Nash equilibrium; this implies no need of coordination to reach an equilibrium*
- *Every Nash equilibrium provides the same utility (payoff) to the players: multiplicity of solutions does not create problems*
- *Nash equilibria are easy to be found in zero sum games*

Symmetric games

Definition

A square matrix $n \times n$ $P = (p_{ij})$ *is said to be antisymmetric provided* $p_{ij} = -p_{ji}$ *for all* $i, j = 1, \dots, n$.
A (finite) zero sum game is said to be fair if the associated matrix is antisymmetric

In fair games there is no favorite player

Fair outcome

Proposition

If $P = (p_{ij})$ is antisymmetric the value is 0 and $\bar{x}$ is an optimal strategy for Pl1 if and only if it is optimal for Pl2

Proof Since

$$x^t P x = (x^t P x)^t = x^t P^t x = -x^t P x$$

then

$$\inf_y x^t P y \leq 0, \qquad \sup_x x^t P y \geq 0$$

$v_1 \leq 0, v_2 \geq 0$. Since they are equal, $v = 0$

If $\bar{x}$ is optimal for the first player, $\bar{x}^t P y \geq 0$ for all y and transposing

$y^t P \bar{x} \leq 0$ for all $y \in \Sigma_n$,

thus $\bar{x}$ is optimal also for the second player, and conversely ■

Finding optimal strategies in a fair game

Need to solve the system of inequalities

$$
\begin{array}{l}
x_1 p_{11} + \cdots + x_n p_{n1} \geq 0 \\
\ldots \\
x_1 p_{1j} + \cdots + x_n p_{nj} \geq 0 \\
\ldots \\
x_1 p_{1m} + \cdots + x_n p_{nn} \geq 0
\end{array} \tag{3}
$$

with the extra conditions:

$$
x_i \geq 0, \qquad \sum_{i=1}^{n} x_i = 1
$$

A proposed exercise

Example

Find the optimal strategies of the following fair game:

$$P = \begin{pmatrix} 0 & 3 & -2 & 0 \\ -3 & 0 & 0 & 4 \\ 2 & 0 & 0 & -3 \\ 0 & -4 & 3 & 0 \end{pmatrix}$$