Zero sum games

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General form

An interesting case is when the game is two player, zero sum

Definition

A two player zero sum game in strategic form is the triplet $(X, Y, f : X \times Y \to \mathbb{R})$

$f(x, y)$ is what Pl1 gets from Pl2, when they play x, y respectively. Thus $g = -f$

Finite game

In the finite case $X = \{1, 2, \dots, n\}$, $Y = \{1, 2, \dots, m\}$ the game is described by a payoff matrix P

Example

$$P = \begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}$$

Pl1 selects row i, Pl2 selects column j.

A different approach to solve them

$$\begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}.$$

Pl1 can guarantee herself to get at least

$$v_1 = \max_i \min_j p_{ij}$$

Pl2 can guarantee himself to pay no more than

$$v_2 = \min_j \max_i p_{ij}$$

$\min_j p_{1j} = 1$, $\min_j p_{2j} = 5$, $\min_j p_{3j} = 0$ $\quad v_1 = 5$
$\min_i p_{i1} = 8$, $\min_j p_{i2} = 5$, $\min_j p_{i3} = 8$, $\quad v_2 = 5$

Rational outcome 5.Rational behavior ($\bar{\imath} = 2, \bar{\jmath} = 2$).

Alternative idea of solution

Suppose $v_1 = v_2 := v$, denote by $\bar{\imath}$ ($\bar{\jmath}$) the row (column) such that $p_{\bar{\imath}j} \geq v$ for all j ($p_{i\bar{\jmath}} \leq v$ for all i).

Then $p_{\bar{\imath}\bar{\jmath}} = v$ and $p_{\bar{\imath}\bar{\jmath}} = v$ is the rational outcome of the game

Remark

$\bar{\imath}$ ($\bar{\jmath}$) is an optimal strategy for Pl1 (for Pl2), because he cannot get more (cannot pay less) than v (since v is the conservative value of the second (first) player)

For arbitrary games

$(X, Y, f : X \times Y \to \mathbb{R})$

The players can guarantee to themselves (almost):

Pl1: $v_1 = \sup_x \inf_y f(x, y)$

PL2: $v_2 = \inf_y \sup_x f(x, y)$

v_1, v_2 are the conservative values of the players

Optimality

Suppose $v_1 = v_2 := v$, strategies $\bar{x}$ and $\bar{y}$ exist such that

$$f(\bar{x}, y) \geq v, \quad f(x, \bar{y}) \leq v$$

for all y and for all x

Then $f(\bar{x}, \bar{y}) = v$ is the rational outcome of the game

$\bar{x}$ is an optimal strategy for Pl1, $\bar{y}$ is an optimal strategy for Pl2

$v_1 \leq v_2$

Proposition

Let X, Y be any sets and let $f : X \times Y \to \mathbb{R}$ be an arbitrary function. Then

$$\sup_x \inf_y f(x,y) \leq \inf_y \sup_x f(x,y)$$

Proof Observe that, for all x, y,

$$\inf_y f(x,y) \leq f(x,y) \leq \sup_x f(x,y)$$

Thus

$$\inf_y f(x,y) \leq \sup_x f(x,y)$$

Since the left hand side of the above inequality does not depend on y and the right hand side on x, the thesis follows ■

In every game $v_1 \leq v_2$, as expected

Equality need not hold

Example

$$P = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

$v_1 = -1$, $v_2 = 1$

Nothing unexpected...

Case $v_1 < v_2$

Finite case: mixed strategies. Game: $n \times m$ matrix P.

Strategy space for Pl1:

$$\Sigma_n = \{x = (x_1, \dots, x_n) : x_i \geq 0, \sum_{i=1}^{n} x_i = 1\}$$

Strategy space for Pl2:

$$\Sigma_m = \{y = (y_1, \dots, y_m) : y_j \geq 0, \sum_{j=1}^{m} y_j = 1\}$$

$$f(x, y) = \sum_{i=1,\dots,n, j=1,\dots,m} x_i y_j p_{ij} = x^t P y$$

The mixed extension of the initial game P: $(\Sigma_n, \Sigma_m, f(x, y) = x^t P y)$

To prove existence of a rational outcome

What must be proved, to have existence of a rational outcome:

1) $v_1 = v_2$
2) there exists $\bar{x}$ fulfilling

$$v_1 = \sup_x \inf_y f(x, y) = \inf_y f(\bar{x}, y)$$

3) there exists $\bar{y}$ fulfilling

$$v_2 = \inf_y \sup_x f(x, y) = \sup_x f(x, \bar{y})$$

In the finite case $\bar{x}$ and $\bar{y}$ fulfilling 1) and 2) always exist; thus existence is equivalent to 1)

The von Neumann theorem

Theorem

A two player, finite, zero sum game as described by a payoff matrix P has a rational outcome.

Finding optimal strategies:Pl1

Pl1 must choose a probability distribution $\Sigma_n \ni x = (x_1, \dots, x_n)$:

$$
\begin{aligned}
&x_1 p_{11} + \cdots + x_n p_{n1} \geq v \\
&\dots \\
&x_1 p_{1j} + \cdots + x_n p_{nj} \geq v \\
&\dots \\
&x_1 p_{1m} + \cdots + x_n p_{nm} \geq v
\end{aligned}
$$

where v must be as large as possible

Finding optimal strategies:Pl2

Pl2 must choose a probability distribution $\Sigma_m \ni y = (y_1, \dots, y_m)$:

$$
\begin{aligned}
&y_1 p_{11} + \cdots + y_m p_{1m} \leq w \\
&\dots \\
&y_1 p_{i1} + \cdots + y_m p_{im} \leq w \\
&\dots \\
&y_1 p_{n1} + \cdots + y_m p_{nm} \leq w
\end{aligned}
$$

where w must be as small as possible

In matrix form

Pl1:

$$\begin{cases} \max_{x,v} v : \\ P^t x \geq v 1_m \\ x \geq 0 \quad \langle 1, x \rangle = 1 \end{cases} \tag{1}$$

Pl2:

$$\begin{cases} \min_{y,w} w : \\ P y \leq w 1_n \\ y \geq 0 \quad \langle 1, y \rangle = 1 \end{cases} \tag{2}$$

Easy to see that these problems are in duality, they are feasible, and the two values agree.

Summarizing

A finite zero sum game has always rational outcome in mixed strategies

The set of optimal strategies for the players is a nonempty closed convex set, the smallest convex set containing a finite number of points, called the extreme points of the set

The outcome, at each pair of optimal strategies, is the common conservative value v of the players

But what about the Nash equilibria of a zero sum game?

Theorem

Let X, Y be (nonempty) sets and $f : X \times Y \to \mathbb{R}$ a function. Then the following are equivalent:

1) *The pair $(\bar{x}, \bar{y})$ fulfills*

$$f(x, \bar{y}) \leq f(\bar{x}, \bar{y}) \leq f(\bar{x}, y) \quad \forall x \in X, \ \forall y \in Y$$

2) *The following conditions are satisfied:*
   (i) $\inf_y \sup_x f(x, y) = \sup_x \inf_y f(x, y)$
   (ii) $\inf_y f(\bar{x}, y) = \sup_x \inf_y f(x, y)$
   (iii) $\sup_x f(x, \bar{y}) = \inf_y \sup_x f(x, y)$

Proof

Proof 1) implies 2). From 1) we get:

$$\inf_y \sup_x f(x,y) \le \sup_x f(x,\bar{y}) = f(\bar{x},\bar{y}) = \inf_y f(\bar{x},y) \le \sup_x \inf_y f(x,y)$$

Since $v_1 \le v_2$ always holds, all above inequalities are equalities

Conversely, suppose 2) holds Then

$$\inf_y \sup_x f(x,y) \overset{(iii)}{=} \sup_x f(x,\bar{y}) \ge f(\bar{x},\bar{y}) \ge \inf_y f(\bar{x},y) \overset{(ii)}{=} \sup_x \inf_y f(x,y)$$

Because of (i), all inequalities are equalities and the proof is complete ■

As a consequence of the theorem

Any $(\bar{x}, \bar{y})$ Nash equilibrium of the zero sum game provides optimal strategies for the players

Any pair of optimal strategies for the players provides a Nash equilibrium for the zero sum game

Thus Nash theorem generalizes von Neumann's

A comment

Remark

Von Neumann approach with conservatives values shows that, in the particular case of the zero sum game:

1) *Players can find their optimal behavior independently for the other players*
2) *Any pair of optimal strategies provides a Nash equilibrium; this implies no need of coordination to reach an equilibrium*
3) *Every Nash equilibrium provides the same utility (payoff) to the players: multiplicity of solutions does not create problems*
4) *Nash equilibria are easy to be found in zero sum games*

Symmetric games

Definition

A square matrix $n \times n$ $P = (p_{ij})$ *is said to be antisymmetric provided* $p_{ij} = -p_{ji}$ *for all* $i, j = 1, \dots, n$. *A (finite) zero sum game is said to be fair if the associated matrix is antisymmetric*

In fair games there is no favorite player

Fair outcome

Proposition

If $P = (p_{ij})$ is antisymmetric the value is 0 and $\bar{x}$ is an optimal strategy for Pl1 if and only if it is optimal for Pl2

Proof Since

$$x^t P x = (x^t P x)^t = x^t P^t x = -x^t P x,$$

$f(x,x) = 0$ for all x thus $v_1 \leq 0, v_2 \geq 0$

Then $v = 0$.

If $\bar{x}$ is optimal for the first player, $\bar{x}^t P y \geq 0$ for all y

Thus $y^t P \bar{x} \leq 0$ for all $y \in \Sigma_n$, and

$\bar{x}$ is optimal for the second player ■

Finding optimal strategies in a fair game

Need to solve the system of inequalities

$$\begin{aligned}
&x_1 p_{11} + \cdots + x_n p_{n1} \geq 0 \\
&\ldots \\
&x_1 p_{1j} + \cdots + x_n p_{nj} \geq 0 \\
&\ldots \\
&x_1 p_{1m} + \cdots + x_n p_{nm} \geq 0
\end{aligned}$$

with the extra conditions:

$$x_i \geq 0, \qquad \sum_{i=1}^{n} x_i = 1$$

A proposed exercise

Example

Find the optimal strategies of the following fair game:

$$P = \begin{pmatrix} 0 & 3 & -2 & 0 \\ -3 & 0 & 0 & 4 \\ 2 & 0 & 0 & -3 \\ 0 & -4 & 3 & 0 \end{pmatrix}$$

Toward Indifference Principle

In the system (with v unknown!)

$$\begin{aligned}
&x_1 p_{11} + \cdots + x_n p_{n1} \geq v \\
&\ldots \\
&x_1 p_{1j} + \cdots + x_n p_{nj} \geq v \\
&\ldots \\
&x_1 p_{1m} + \cdots + x_n p_{nm} \geq v
\end{aligned}$$

when a strict inequality is possible?

Suppose $\bar{x}$ is optimal for Pl1 and

$\bar{x}_1 p_{1j} + \cdots + \bar{x}_n p_{nj} > v$.

Then Pl2 never plays column j.

Otherwise Pl1 would get more than v playing $\bar{x}$.

The Principle

There is a nonempty set of indices $J_1 = \{j_1, \dots, j_k\}$ such that

$$x_1 p_{1j_1} + \dots + x_n p_{nj_1} = x_1 p_{1j_2} + \dots + x_n p_{nj_2} = \dots = x_1 p_{1j_k} + \dots + x_n p_{nj_k}$$

and

$$x_1 p_{1j_1} + \dots + x_n p_{nj_1} > x_1 p_{1j} + \dots + x_n p_{nj}$$

for all $j \notin J_1$

J_1 is the set of columns played with positive probability by Pl2 at some optimal strategy

Also true: if $j \notin J_1$ there exists an optimal strategy for Pl1 providing her a payoff $> v$ against column j