Game Theory

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General form

An interesting case is when the game is two player, zero sum.

Definition

A two player zero sum game in strategic form is the triplet $(X, Y, f : X \times Y \to \mathbb{R})$

$f(x, y)$ is what Pl1 gets from Pl2, when they play x, y respectively. Thus $g = -f$.

Finite game

In the finite case $X = \{1, 2, \dots, n\}$, $Y = \{1, 2, \dots, m\}$ the game is described by a payoff matrix P.

Example

$$P = \begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}.$$

Pl1 selects row i, Pl2 selects column j.

A different approach to solve them

$$\begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}.$$

Pl1 can guarantee herself

$$v_1 = \max_i \min_j p_{ij}$$

Pl2 can guarantee himself

$$v_2 = \min_j \max_i p_{ij}$$

$\min_j p_{1j} = 1$, $\min_j p_{2j} = 5$, $\min_j p_{3j} = 0$ $\quad v_1 = 5$

$\min_i p_{i1} = 8$, $\min_j p_{i2} = 5$, $\min_j p_{i3} = 8$, $\quad v_2 = 5$
Rational outcome 5.Rational behavior ($\bar{i} = 2, \bar{j} = 2$).

Alternative idea of solution

Suppose $v_1 = v_2 := v$, denote by $\bar{\imath}$ ($\bar{\jmath}$) the row (column) such that $p_{\bar{\imath}j} \geq v$ for all j ($p_{i\bar{\jmath}} \leq v$ for all i).

Then $p_{\bar{\imath}\bar{\jmath}} = v$ and $p_{\bar{\imath}\bar{\jmath}} = v$ is the rational outcome of the game.

$\bar{\imath}$ is an optimal strategy for Pl1, $\bar{\jmath}$ is an optimal strategy for Pl2.

Generalizing

Consider the game $(X, Y, f : X \times Y \to \mathbb{R})$.
The players can guarantee to themselves (almost):

Pl1: $v_1 = \sup_x \inf_y f(x, y)$

PL2: $v_2 = \inf_y \sup_x f(x, y)$

v_1, v_2 are the conservative values of the players

Optimality

Suppose $v_1 = v_2 := v$, strategies $\bar{x}$ and $\bar{y}$ exist such that

$$f(\bar{x}, y) \geq v, \quad f(x, \bar{y}) \leq v$$

for all y and for all x.

Then $f(\bar{x}, \bar{y}) = v$ is the rational outcome of the game.

$\bar{x}$ is an optimal strategy for Pl1, $\bar{y}$ is an optimal strategy for Pl2.

$v_1 \leq v_2$.

Proposition

Let X, Y be any sets and let $f : X \times Y \to \mathbb{R}$ be an arbitrary function. Then

$$\sup_x \inf_y f(x,y) \leq \inf_y \sup_x f(x,y).$$

Proof Observe that, for all x, y,

$$\inf_y f(x,y) \leq f(x,y) \leq \sup_x f(x,y).$$

Thus

$$\inf_y f(x,y) \leq \sup_x f(x,y)$$

Since the left hand side of the above inequality does not depend on y and the right hand side on x, the thesis follows. ■

i.e. in every game $v_1 \leq v_2$.

Another example

Example

$$P = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

$v_1 = -1$, $v_2 = 1$

Nothing unexpected...

Case $v_1 < v_2$

Finite case: mixed strategies. Game: $n \times m$ matrix P.

Strategy space for Pl1:

$$\Sigma_n = \{x = (x_1, \dots, x_n) : x_i \geq 0, \sum_{i=1}^{n} x_i = 1\}$$

Strategy space for Pl2:

$$\Sigma_m = \{y = (y_1, \dots, y_m) : y_j \geq 0, \sum_{j=1}^{m} y_j = 1\}$$

$$f(x, y) = \sum_{i=1,\dots,n, j=1,\dots,m} x_i y_j p_{ij} = \langle x, Py \rangle = \langle P^t x, y \rangle = x^t P y$$

The mixed extension of the initial game P: $(\Sigma_n, \Sigma_m, f(x, y) = x^t P y)$

To prove existence of a rational outcome

What must be proved, to have existence of a rational outcome:

1) $v_1 = v_2$;
2) there exists $\bar{x}$ fulfilling

$$v_1 = \sup_x \inf_y f(x, y) = \inf_y f(\bar{x}, y)$$

3) there exists $\bar{y}$ fulfilling

$$v_2 = \inf_y \sup_x f(x, y) = \sup_x f(x, \bar{y})$$

The von Neumann theorem

Theorem

A two player, finite, zero sum game as described by a payoff matrix P has a rational outcome.

The players have optimal strategies and $v := v_1 = v_2$ is what PL1 receives from Pl2.

Finding optimal strategies:Pl1

Pl1 must choose a probability distribution $\Sigma_n \ni x = (x_1, \dots, x_n)$:

$$\begin{aligned} & x_1 p_{11} + \cdots + x_n p_{n1} \geq v \\ & \dots \\ & x_1 p_{1j} + \cdots + x_n p_{nj} \geq v \\ & \dots \\ & x_1 p_{1m} + \cdots + x_n p_{nm} \geq v, \end{aligned} \tag{1}$$

where v must be as large as possible.

Finding optimal strategies:Pl2

Pl2 must choose a probability distribution $\Sigma_m \ni y = (y_1, \dots, y_m)$:

$$\begin{aligned} & y_1 p_{11} + \cdots + y_m p_{1m} \leq w \\ & \ldots \\ & y_1 p_{i1} + \cdots + y_m p_{im} \leq w \\ & \ldots \\ & y_1 p_{n1} + \cdots + y_m p_{nm} \leq w, \end{aligned} \tag{2}$$

where w must be as small as possible.

In matrix form

Pl1:

$$\begin{cases} \max_{x,v} v : \\ P^t x \geq v 1_m \\ x \geq 0 \quad \langle 1, x \rangle = 1 \end{cases} . \tag{3}$$

Pl2:

$$\begin{cases} \min_{y,w} w : \\ P y \leq w 1_n \\ y \geq 0 \quad \langle 1, y \rangle = 1 \end{cases} . \tag{4}$$

Easy to see that these problems are in duality, they are feasible, and the two values agree.

Summarizing

A finite zero sum game has always rational outcome in mixed strategies

The set of optimal strategies for the players is a nonempty closed convex set

The outcome, at each pair of optimal strategies, is the common conservative value v of the players