The Nash model

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Summary of the slides

1. The strategic form of the game
2. Nash equilibrium profile
3. Existence of Nash equilibrium profiles
4. Existence in mixed strategies for finite game
5. Best reply multifunction
6. Indifference principle
7. Games with many players
8. Braess paradox
9. El Farol bar
10. Congestion games
11. Duopoly models

Definition of non cooperative game

Definition

A two player noncooperative game in strategic form is
$(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$

X, Y are the strategy sets of the players, f, g their utility functions.

Equilibrium

A Nash equilibrium profile for the $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$ is a pair $(\bar{x}, \bar{y}) \in X \times Y$ such that:

- $f(\bar{x}, \bar{y}) \geq f(x, \bar{y})$ for all $x \in X$
- $g(\bar{x}, \bar{y}) \geq g(\bar{x}, y)$ for all $y \in Y$

A Nash equilibrium profile is a joint combination of strategies, stable w.r.t. unilateral deviations of a single player

The new rationality paradigm

Observe: new definition of rationality

Need to compare with former concepts

1) Suppose $\bar{x}$ is a (weakly) dominant strategy for Pl1:

$f(\bar{x}, y) \geq f(x, y)$ for all x, y.

Then, if $\bar{y}$ maximizes the function $y \rightarrowtail g(\bar{x}, y)$,

$(\bar{x}, \bar{y})$ is a NEp.

Nash equilibria in games with perfect information

2) Backward induction provides a Nash equilibrium profile for a game of perfect information. Is it possible that in games of perfect information there are more equilibria than that one(s) provided by backward induction?

Example

Player 1 must claim for himself $x \in [0,1]$*. Player 2 can either accept* $(1-x)$ *or decline. If she declines both players get 0, otherwise utilities are* $(x, 1-x)$

Backward induction provides strategies

- Propose $x = 1$ for the first player
- Accept any offer for the second player

The outcome is $(1,0)$: the first player keeps all money.

On the contrary, any outcome $(x, 1-x)$ is the result of a NE profile (prove it!).

Existence of Nash equilibria

Denote by BR_1 the following multifunction:

$$BR_1 : Y \to X : \quad BR_1(y) = \mathrm{Arg\ Max}\ \{f(\cdot, y)\}$$

$$BR_2 : X \to Y : \quad BR_2(x) = \mathrm{Arg\ Max}\ \{g(x, \cdot)\}$$

and

$$BR : X \times Y \to X \times Y : \quad BR(x, y) = (BR_1(y), BR_2(x)).$$

Then $(\bar{x}, \bar{y})$ is a Nash equilibrium profile for the game if and only if

$$(\bar{x}, \bar{y}) \in BR(\bar{x}, \bar{y}).$$

Thus existence of a Nash equilibrium profile in a game is equivalent to existence of a fixed point for the Best Reaction Multifunction.

The Nash theorem

Theorem

Given the game $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$, *suppose:*

1. X *and* Y *are compact convex subsets of some Euclidean space*
2. f, g *continuous*
3. $x \mapsto f(x, y)$ *is (quasi) concave for all* $y \in Y$
4. $y \mapsto g(x, y)$ *is (quasi) concave for all* $x \in X$

Then the game has at least one Nash equilibrium profile.

Quasi concavity for a real valued function h means that the sets

$$h_a = \{z : h(z) \geq a\}$$

are convex for all a (maybe empty for some a).

Finite games: notation

Suppose the sets of the strategies of the players are finite, $\{1, \dots, n\}$ for the first player, $\{1, \dots, m\}$ for the second player. Then the game can be represented by the bimatrix

$$\begin{pmatrix} (a_{11}, b_{11}) & \dots & (a_{1m}, b_{1m}) \\ \dots & \dots & \dots \\ (a_{n1}, b_{n1}) & \dots & (a_{nm}, b_{nm}) \end{pmatrix}$$

where a_{ij} (b_{ij}) is the utility of the row (column)player when row plays strategy i and column strategy j.

Denote by (A, B) such a game.

Finite games

Corollary

A finite game (A, B) admits always a Nash equilibrium profile in mixed strategies

In this case X and Y are simplexes, $f(x, y) = x^t Ay, g(x, y) = x^t By$, and thus the assumption of the theorem are fulfilled.

Expliciting utilities:

$$f(x, y) = \sum_{i=1,\dots,n, j=1,\dots,m} x_i y_j a_{ij}, \quad g(x, y) = \sum_{i=1,\dots,n, j=1,\dots,m} x_i y_j b_{ij}$$

Remark

Once fixed the strategies of the other players, the utility function of one player is linear in its own variable

Finding Nash equilibria

The game:

$$\begin{pmatrix} (1,0) & (0,3) \\ (0,2) & (1,0) \end{pmatrix}$$

Pl1 playing $(p, 1-p)$, PL2 playing $(q, 1-q)$:

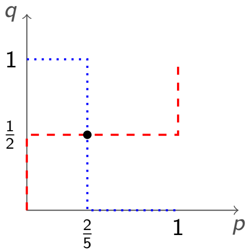
$$f(p,q) = pq + (1-p)(1-q) = p(2q-1) - q + 1$$

$$g(p,q) = 3p(1-q) + 2(1-p)q = q(2-5p) + 3p$$

The best reply multifunctions

$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad 0 \leq q \leq \frac{1}{2} \\ p \in [0,1] & \text{if} \quad q = \frac{1}{2} \\ p = 1 & \text{if} \quad q > \frac{1}{2} \end{cases}$$

$$BR_2(p) = \begin{cases} q = 1 & \text{if} \quad 0 \leq p \leq \frac{2}{5} \\ q \in [0,1] & \text{if} \quad p = \frac{2}{5} \\ q = 0 & \text{if} \quad p > \frac{2}{5} \end{cases}$$



Equalities and inequalities to find NEp

Remark

Suppose $(\bar{x}, \bar{y})$ is a NE in mixed strategies. Suppose spt $\bar{x} = \{1, \dots, k\}$[1], spt $\bar{y} = \{1, \dots, l\}$, and $f(\bar{x}, \bar{y}) = v$. Then it holds:

$$\begin{cases} a_{11}\bar{y}_1 + a_{12}\bar{y}_2 + \dots + a_{1l}\bar{y}_l & = v \\ \dots & = v \\ a_{k1}\bar{y}_1 + a_{k2}\bar{y}_2 + \dots + a_{kl}\bar{y}_l & = v \\ a_{(k+1)1}\bar{y}_1 + a_{(k+1)2}\bar{y}_2 + \dots + a_{(k+1)l}\bar{y}_l & \leq v \\ \dots & \leq v \\ a_{n1}\bar{y}_1 + a_{n2}\bar{y}_2 + \dots + a_{nl}\bar{y}_l & \leq v \end{cases}$$

The above relations are due the fact that rows used with positive probability must be all optimal (and thus they all give the same expected value), while the other ones are suboptimal

[1] spt $\bar{x} = \{i : \bar{x}_i > 0\}$

An example

In the following game, find a, b such that there is a Nash equilibrium with support the first two rows for the first player and the columns 2 and 3 for the second one;

$$\begin{pmatrix} (2,2) & (a,3) & (3,3) \\ (4,0) & (3,4) & (5,b) \\ (2,3) & (5,2) & (4,26) \end{pmatrix},$$

The system to impose, about the first player:

$$aq + 3 - 3q = 3q + 5 - 5q, \qquad 3q + 5 - 5q \geq 5q + 4 - 4q$$

providing the conditions

$$q = \frac{2}{a-1}, \qquad q \leq \frac{1}{3}.$$

For consistency, this implies $a \geq 7$. For the second player the first column is strictly dominated, and it must be $b = 4$ (otherwise one column dominates the other one) and in this case every $p \in (0,1)$ works.

Full support

The above system of equalities/inequalities simplifies a lot if one looks for fully mixed[2] Nash equilibria.

Suppose $(\bar{x}, \bar{y})$ is such a Nash equilibrium profile. Then it holds that

$$a_{i1}\bar{y}_1 + a_{i2}\bar{y}_2 + \cdots + a_{im}\bar{y}_m = a_{k1}\bar{y}_1 + a_{k2}\bar{y}_2 + \cdots + a_{km}\bar{y}_m$$

for all $i, k = 1, \dots n$, and similarly

$$b_{1r}\bar{x}_1 + b_{2r}\bar{x}_2 + \cdots + b_{nr}\bar{x}_n = b_{1s}\bar{x}_1 + b_{2s}\bar{x}_2 + \cdots + b_{ns}\bar{x}_n$$

for all $r, s = 1, \dots m$ with the further conditions

$$p_j, q_j \geq 0, \quad \sum_{i=1}^{n} p_i = 1, \quad \sum_{j=1}^{m} q_j = 1$$

In this case we speak about Indifference principle.

[2] This means that all rows/columns are played with positive probabilities

Brute force algorithm

1. Guess the supports of the equilibria $spt(\bar{x})$ and $spt(\bar{y})$
2. Ignore the inequalities and find x, y, v, w by solving the linear system of $n + m + 2$ equations

$$\begin{cases} \sum_{i=1}^{n} x_i = 1 & \\ \sum_{j=1}^{m} a_{ij} y_j = v & \text{for all } i \in spt(\bar{x}) \\ x_i = 0 & \text{for all } i \notin spt(\bar{x}) \end{cases}$$

$$\begin{cases} \sum_{j=1}^{m} y_j = 1 & \\ \sum_{i=1}^{n} b_{ij} x_i = w & \text{for all } j \in spt(\bar{y}) \\ y_j = 0 & \text{for all } j \notin spt(\bar{y}) \end{cases}$$

3. Check whether the ignored inequalities are satisfied. If $x_i \geq 0, y_j \geq 0, \sum_{j=1}^{m} a_{ij} y_j \leq v$ and $\sum_{i=1}^{n} b_{ij} x_i \leq w$ then Stop: we have found a mixed equilibrium profile. Otherwise, go back to step 1 and try another guess of the supports.

Lemke-Howson Algorithm

Enumerating all the possible supports in the brute force algorithm quickly becomes computationally prohibitive: there are potentially $(2^n - 1)(2^m - 1)$ options!

For $n \times n$ games the number of combinations grow very quickly

n	# of potential supports
2	9
3	49
4	225
5	961
10	1.046.529
20	1.099.509.530.625

Lemke-Howson proposed a more efficient algorithm... though still with exponential running time in the worst case.

General strategic games

Consider an n-player game with strategy sets X_i and payoffs $f_i : X \to \mathbb{R}$ with $X = \prod_{j=1}^{n} X_j$.

Notation:
if $x = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$ is a strategy profile, denote by x_{-i} the vector $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ and write also $x = (x_i, x_{-i})$.

Then: $\bar{x} = (\bar{x}_i)_{i=1}^{n}$ is a NE p. if and only if for each player $i = 1, \dots, n$ we have $\bar{x}_i \in BR_i(\bar{x}_{-i})$.

The Nash theorem

Theorem

Given a n-player game with strategy sets X_i and payoff functions $f_i : X \to \mathbb{R}$ where $X = \prod_{i=1}^{n} X_i$. Suppose:

- *each X_i is a closed bounded convex subset in a finite dimensional space $\mathbb{R}^{d_i}$*
- *each $f_i : X \to \mathbb{R}$ is continuous*
- *$x_i \mapsto f_i(x_i, x_{-i})$ is a (quasi) concave function for each fixed $x_{-i} \in X_{-i}$*

Then the game admits at least one Nash equilibrium profile.

Mixed equilibria for n-player finite games

Consider an n-person finite game with strategy sets A_i and payoffs $f_i(a_1, \dots, a_n)$. In the *mixed extension* each player i choses a probability distribution $x^i \in \Sigma_{A_i}$, that is to say, $x^i_{a_i} \geq 0$ for all $a_i \in A_i$ and $\sum_{a_i \in A_i} x^i_{a_i} = 1$.
Denote $A = \prod_{i=1}^n A_i$ the set of pure strategy profiles. The probability of observing an outcome $(a_1, \dots, a_n) \in A$ is the product $\prod_{i=1}^n x^i_{a_i}$ and the *expected* payoffs are:

$$\bar{f}_i(x^1, \dots, x^n) = \sum_{(a_1,\dots,a_n)\in A} f_i(a_1, \dots, a_n) \prod_{j=1}^n x^j_{a_j} = \sum_{a_i \in A_i} x^i_{a_i} u_i(a_i, x^{-i})$$

$$u_i(a_i, x^{-i}) = \sum_{a_j \in A_j, j \neq i} f_i(a_1, \dots, a_n) \prod_{j \neq i} x^j_{a_j}$$

Corollary

Every n-player finite game has at least one Nash equilibrium profile in mixed strategies.

First example: the Braess paradox

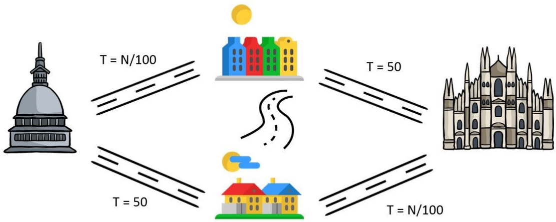


Figure: Commuting

4.000 people travel form one city to another one. Every player wants to minimize time. N is the number of people driving in the corresponding road

What are the Nash equilibria? What happens if the North-South street between the two small cities is made available to cars and time to travel on it is 5 minutes?

El Farol bar

In Santa Fe there are 500 young people, happy to go to the El Farol bar. More people in the bar, happier they are, till they reach 300 people. They can also choose to stay at home. So utility function can be assumed to be 0 if they stay at home, $u(x) = x$ if $x \leq 300$, $u(x) = 300 - x$ if $x > 300$.

Congestion games have always (pure) Nash equilibria, necessarily asymmetric!

A mixed symmetric Nash equilibrium profile is present in this case.

Duopoly models

Two firms choose quantities of a good to produce. Firm 1 produces quantity q_1, firm 2 produces quantity q_2, the unitary cost of the good is $c > 0$ for both firms. A quantity $a > c$ of the good saturates the market. The price $p(q_1, q_2)$ is

$$p = \max\{a - (q_1 + q_2), 0\}$$

Payoffs:

$$u_1(q_1, q_2) = q_1 p(q_1, q_2) - cq_1 = q_1(a - (q_1 + q_2)) - cq_1,$$
$$u_2(q_1, q_2) = q_2 p(q_1, q_2) - cq_2 = q_2(a - (q_1 + q_2)) - cq_2.$$

The monopolist

Suppose $q_2 = 0$.

Firm 1 maximizes $u(q_1) = q_1(a - q_1) - cq_1$.

$$q_M = \frac{a-c}{2}, \quad p_M = \frac{a+c}{2} \quad u_M(q_M) = \frac{(a-c)^2}{4}$$

The duopoly

The utility functions are strictly concave and non positive at the endpoints of the domain, thus the first derivative must vanish:

$$a - 2q_1 - q_2 - c = 0, \qquad a - 2q_2 - q_1 - c = 0,$$

$$q_i = \frac{a-c}{3}, \quad p = \frac{a+2c}{3} \quad u_i(q_i) = \frac{(a-c)^2}{9}.$$

The case with a leader

One firm, the Leader, announces its strategy, and the other one, the Follower, acts taking for granted the announced strategy of the Leader.

$$\bar{q}_2(q_1) = \frac{a - q_1 - c}{2}.$$

The Leader maximizes

$$u_1(q_1, \frac{a - q_1 - c}{2})$$

$$\bar{q}_1 = \frac{a-c}{2},\ \bar{q}_2 = \frac{a-c}{4},\ u_1(\bar{q}_1, \bar{q}_2) = \frac{(a-c)^2}{8},\ u_2(\bar{q}_1, \bar{q}_2) = \frac{(a-c)^2}{16}$$

$$p = \frac{a + 3c}{4}$$

Comparing the three cases 1

Monopoly

$$q_M = \frac{a-c}{2}, \quad p_M = \frac{a+c}{2} \quad u_M(q_M) = \frac{(a-c)^2}{4}$$

Duopoly

$$q_i = \frac{a-c}{3}, \quad p = \frac{a+2c}{3} \quad u_i(q_i) = \frac{(a-c)^2}{9}.$$

Leader

$$\bar{q}_1 = \frac{a-c}{2},\ \bar{q}_2 = \frac{a-c}{4},\ u_1(\bar{q}_1, \bar{q}_2) = \frac{(a-c)^2}{8},\ u_2(\bar{q}_1, \bar{q}_2) = \frac{(a-c)^2}{16}$$

$$p = \frac{a+3c}{4}$$

Comparing the three cases 2

Making a comparison with the case of a monopoly, we see that:

- the price is lower in the duopoly case;
- the total quantity of product in the market is superior in the duopoly case;
- the total payoff of the two firms is less than the payoff of the monopolist.

In particular, the two firm could consider the strategy of equally sharing the payoff of the monopolist, but this is not a NE profile! The result shows a very reasonable fact, the consumers are better off if there is no monopoly.