Extensive form games

Roberto Lucchetti

Politecnico di Milano

Summary of the slides

1. Extensive form game
2. Perfect information
3. The tree of the game
4. Backward induction
5. Zermelo theorem
6. Different (types of) solutions
7. Combinatorial games
8. The Nim game and Bouton theorem
9. Strategies

Extensive form

Three politicians must vote if to raise or not their salaries. Vote is public and in sequence. Their first option is to have an increase in the salaries, but they would like to vote against.

Main features

1. The moves are in sequence
2. Every possible situation is known to the players, at any time they know the whole past history, and the possible developments

Games with perfect information

How can we represent them?

How can we solve them?
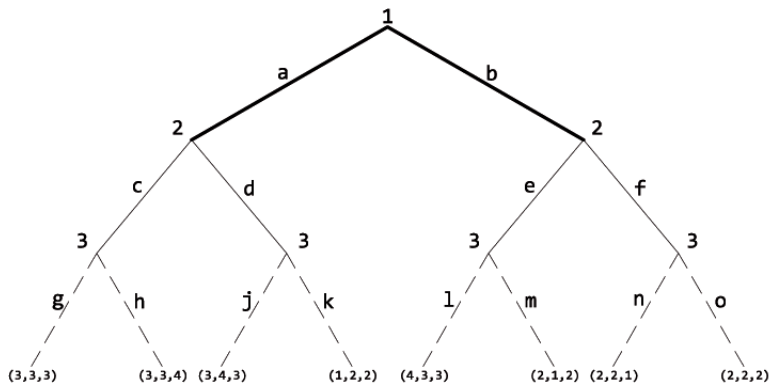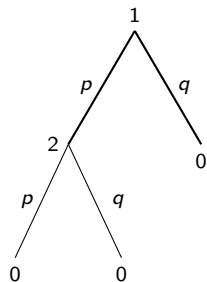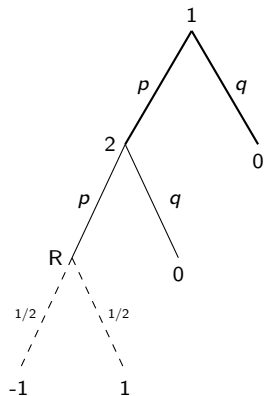
The tree

The tree

Figure: The game of the three politicians

A game with chance

The players must decide in sequence whether to play or not. If both decide to play, a coin is tossed, and the first one wins if the coin shows head, otherwise is the second to win.

Directed graphs

Definition

A *finite directed graph* is a pair (V, E) where

1. V is a finite set, called the set of vertices
2. $E \subset V \times V$ is a set of ordered pairs of vertices called the set of the (directed) edges

Definition

A *path* from a vertex v_1 to a vertex v_{k+1} is a finite sequence of vertices-edges $v_1, e_1, v_2, \ldots, v_k, e_k, v_{k+1}$ such that $e_i \neq e_j$ if $i \neq j$ and $e_j = (v_j, v_{j+1})$. k is called the *length* of the path

Tree

Definition

A *tree* is a triple (V, E, x_0) where (V, E) is an oriented graph and x_0 is a vertex in V such that there is a unique path from x_0 to x, where x is any vertex in V

Definition

A *child* of a vertex v is any vertex x such that $(v, x) \in E$. A vertex is a *leaf* if it has no children, the vertex x follows the vertex v if there is a path from v to x.

The tree of a game

In order to have a complete presentation of a game with a tree, we need to add some requirements.

Definition

An *Extensive form Game with perfect information* *is constituted by*

1. *A finite set $N = \{1, \ldots, n\}$[1] of players*

2. *A game tree (V, E, x_0)*

3. *A partition made by sets $P_1, P_2, \ldots, P_{n+1}$ of the vertices which are not leaves[2]*

4. *A probability distribution, for each vertex in P_{n+1} defined on the edges going from the vertex to its children*

5. *An n-dimensional vector attached to each leaf*

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[1]n will denote the cardinality of the set N.
[2]Some of these $n + 1$ sets could be empty

Comments

1. The set P_i, for $i \leq n$, is the set of the nodes v where Player i must choose a child of v, representing a possible move form him at v

2. P_{n+1} is the set of the nodes where a chance move is present. P_{n+1} can be empty

3. Also some P_i could be empty: this corresponds to the possible situation when somebody cannot take a decision, but the choice of the final outcome can affect him

4. When P_{n+1} is empty (i.e. no chance moves are present in the game) the players need to have only preferences on the leaves: a utility function is not required

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Solving the game

We use the rationality axioms:

1. What one player does in positions leading to leaves can be determined by decision theory (property 5)
2. This is known to all other players (property 4) and this information can be used used by them
3. Thus players moving at vertices going to leaves can use decision theory (property 5)
4. This is known to all other players (property 4) and this information can be used used by them
5. . . .
6. The player starting the game uses decision theory to conclude

Backward induction

Definition

Define Length of the game as the length of the longest path in the game.

Decision theory allows solving the games of length 1

Axiom 4 allows solving a game of length $i + 1$ if the games of length at most i are solved

Thus we can solve games of any finite length

This method is called the method of backward induction

The first rationality theorem

Theorem

The rational outcomes of a finite, perfect information game are those given by the procedure of the backward induction.

Observe: this method can be applied since every vertex v of the game is the root of a new game, made by all followers of v in the initial game. This game is called a subgame of the original one

Multiple solutions
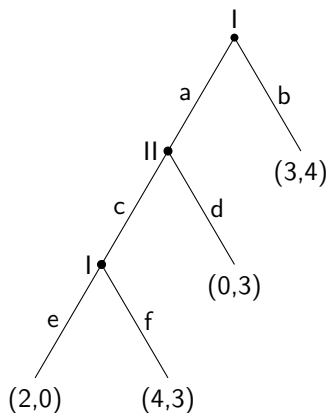
The outcomes obtained by backward induction are: $(4, 3)$ and $(3, 4)$.
Thus uniqueness is not guaranteed.

The chess (Zermelo) theorem

Theorem

In the game of chess one and only one of the following alternatives holds:

1. *The white has a way to win, no matter what the black does*
2. *The black has a way to win, no matter what the white does*
3. *The white has a way to force at least a draw, no matter what the black does, and the same holds for the black*

Main question: is the above a true theorem?

Why is it impossible to say more than that?

An interesting proof of the chess theorem

Backward induction can be applied to the chess game. Thus the Zermelo theorem can be seen as a consequence of applying backward induction. However the argument is not complete, since uniqueness of the outcome is not guaranteed. Later we shall see that in general win-draw-loose games even with multiple outcomes the result must be, as intuitive, always the same. For the moment here we provide an alternative proof of the theorem.

Suppose the length of the game is $2K$ so each player has K choices to make. Call a_i the move of the White at her i-th stage and b_i that one of the Black.

The first alternative in the chess theorem can be expressed as

$$\exists a_1 : \forall b_1 \exists a_2 : \forall b_2 \ldots \exists a_K : \forall b_K \Rightarrow \text{white wins}$$

Now suppose this is not true. Then

$$\forall a_1 \exists b_1 : \forall a_2 \exists b_2 : \ldots \forall a_K \exists b_K : \Rightarrow \text{white does not win}$$

But this means exactly that Black has the possibility to get at least a draw

Cont'd

Summarizing If White does not have the way to win no matter what Black does, then Black has the possibility to get at least the draw

Symmetrically if Black does not have the way to win no matter what White does, then White has the possibility to get at least the draw

Thus if the first and the second alternatives in the chess theorem are not true, necessarily the third one is true!

Extending Zermelo theorem

The Zermelo theorem applies to every finite game of perfect information where the possible result is either the victory of one player or a tie. Thus, if the tie is not allowed, the following Corollary holds

Corollary

Suppose to have a finite perfect information game with two players, and the outcomes are the victory of either player. Then one and only one of the following alternative holds:

1. *The first player has a way to win, no matter what the second one does*

2. *The second player has a way to win, no matter what the first does*

Different types of solutions

Does the chess theorem say anything useful?

Very weak solution

The game has a single rational outcome, inaccessible like for the Chess

Weak solution

The outcome of the game is known, but how to get it is not

Solution

It is possible to provide an algorithm to find a solution
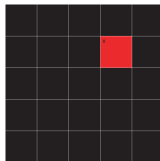
Chomp

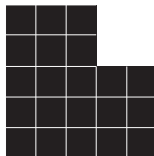Figure: The first player removes the red square

Figure: Now it is up to the second player to move

The player taking the most left-down square looses the game

Chomp
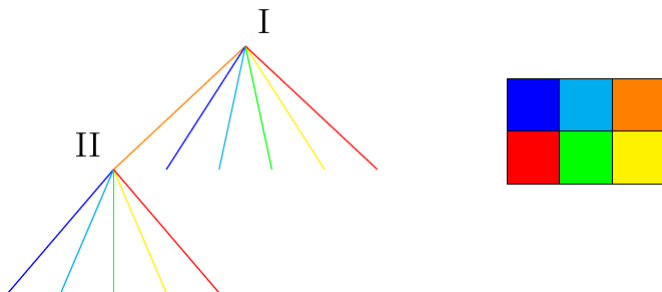
Figure: Edges are coloured according to the chosen square

Suppose the second wins. Then he has a winning action after the choice of the orange edge. Suppose the winning move for him is to choose the green edge. This means that the player starting at the node following the green edge will loose. But the tree starting at that node is exactly the same tree starting at the green edge relative to the first player. Thus the game starting at the node following the green edge going out from the root of the tree (move of the first player) has the starting player (i.e. the second player) loosing. Contradiction!

Finding the solution

Definition

An *impartial combinatorial game* is a game such that

1. There are two players moving in alternate order
2. There is a finite number of *positions* in the game
3. Both players have the same rules to follow
4. The game ends when no moves are possible anymore
5. Chance is not present in the game
6. In the *classical* version the winner is the player leaving the other player with no available moves, in the *misère* version the opposite

Examples of combinatorial games

1. k piles of cards. At her turn the player takes as many cards as she wants (at least one!) from one and only one pile

2. k piles of cards. At her turn the player takes as many cards as she wants (at least one!) from not more than $j < k$ piles

3. k cards in a row. At her turn the player takes either j_1 or ... or j_l objects

In the first two cases the positions are $(n_1, \ldots, n_k)$ where n_i is a non negative integer for all i. In the last examples positions can be seen as all non negative integers smaller or equal to k.

The idea to solve these games

Partition the set of all possible positions into two sets:

1. P-positions
2. N-positions

Rules:

1. Terminal positions are P-positions[3]
2. From a P-position only N-positions are available
3. From an N-position it is possible to go to a P-position

Player beginning at an N-position wins!

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[3]Terminal means that the player does not have a possible move (classical version)

The Nim game

Nim game is defined as $(n_1, \ldots, n_k)$ where for all i n_i is a positive integer. A player at her turn has to take one (and only one) n_i and substitute it with $\hat{n}_i < n_i$. The winner is the player arriving to the position $(0, \ldots, 0)$.

Meaning: taking away cards form one pile. Goal: to clear the table.

A new operation on the non negative integers

Define an operation $\oplus$ on $\mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}$ in the following way: for $n_1, n_2 \in \mathbb{N}$

1. Write n_1, n_2 in binary form (notation for n in binary form $[n]_2$)
2. Write the sum $[n_1]_2 \oplus [n_2]_2$ in binary form where $\oplus$ is the (usual) sum but without carry
3. The result is the obtained number, written in binary form

An example

Example

The $\oplus$ operation applied to 1,2,4, and 1.

$$
\begin{array}{rcccc}
[1]_2 & = & 0 & 0 & 1 \\
[2]_2 & = & 0 & 1 & 0 \\
[4]_2 & = & 1 & 0 & 0 \\
[1]_2 & = & 0 & 0 & 1 \\
\hline
[6]_2 & = & 1 & 1 & 0 \\
\end{array}
$$

$n_1 \oplus n_2 \oplus n_3 \oplus n_4 = 6$

The group

Definition

A nonempty set A with an operation $\cdot$ on it is called a group provided:

1. *for $a, b \in A$ the element $a \cdot b \in A$*

2. *$\cdot$ is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$*

3. *there is an element (which will result unique, and called identity) e such that $a \cdot e = e \cdot a = a$ for all $a \in A$*

4. *for every $a \in A$ there is $b \in A$ such that $a \cdot b = b \cdot a = e$. Such an element is unique and called inverse of a*

If $a \cdot b = b \cdot a$ for all $a, b \in A$ the group is called abelian

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Examples and properties

Example

Examples of groups are

1. *The integers with operation $\cdot$ being the usual sum*
2. *The non null reals with operation $\cdot$ being the usual product*
3. *The $n \times n$ matrices with non null determinant row by column product*

Proposition

Let $(A, \cdot)$ be a group. Then the cancelation law holds:

$$a \cdot b = a \cdot c \implies b = c$$

The Nim group

Proposition

The set of the natural numbers with $\oplus$ in an abelian group

Proof The identity element is 0, the inverse of n is n itself. Associativity and commutativity of $\oplus$ are easy. ■

The cancelation law holds: $n_1 \oplus n_2 = n_1 \oplus n_3$ implies $n_2 = n_3$.

The Bouton theorem

Theorem (**Bouton**)

A $(n_1, n_2, \ldots, n_k)$ position in the Nim game is a P-position if and only if $n_1 \oplus n_2 \oplus \ldots \oplus n_k = 0$.

Proof

- Terminal states are P-positions This is obvious, the only terminal position is $(0, \ldots, 0)$
- Positions such that $n_1 \oplus n_2 \oplus \ldots \oplus n_N = 0$ go only to positions with Nim sum different from zero Suppose instead the new position is $(n'_1, n_2, \ldots, n_N)$ and $n'_1 \oplus n_2 \oplus \ldots \oplus n_N = 0 = n_1 \oplus n_2 \oplus \ldots \oplus n_N$; then by the cancelation law $n'_1 = n_1$, which is impossible
- Positions with non null nim sum can go to a position with null Nim sum. Let $z := n_1 \oplus n_2 \oplus \ldots \oplus n_N \neq 0$. Take a pile having 1 in the most left column where the expansion of z has 1, put there 0 and go right, leaving unchanged a digit corresponding to a 0 in the expansion of the sum, 1 otherwise. Easy to check that the so obtained number is smaller than the previous one. ∎

An example

Example

From

$$
\begin{array}{ccc}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{array}
$$

go to

$$
\begin{array}{ccc}
0 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{array}
$$

Observe: there are three initial good moves.

Conclusions

Games with perfect information can be "solved" by using backward induction

However backward induction is a concrete solution method only for very simple games, because of limited rationality.

According to conclusions we can reach different level of solutions:

1. Very weak solutions: not even the outcome is predictable (chess...)
2. Weak solutions: a logical argument provides the outcome, but how to reach it is not known (chomp, in general)
3. Solutions : categories of games where it is possible to produce the way to arrive to the rational outcome

Strategies

In Backward induction a move must be specified at any node. P_i is the set of the nodes where player i is called to make a move.

Definition

A *pure strategy* for player i is a function defined on the set P_i, associating to each node v in P_i a child w, or equivalently the edge (v, x).
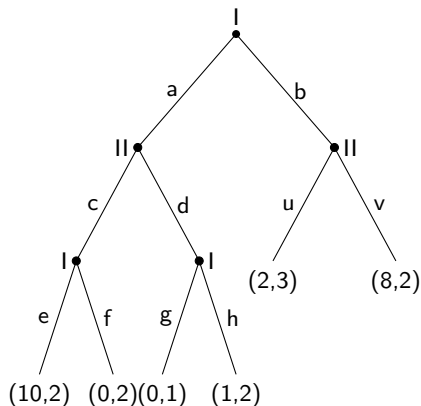
A *mixed strategy* is a probability distribution on the set of the pure strategies.

When a player has n pure strategies, the set of her mixed strategies is

$$\Sigma_n := \{p = (p_1, \ldots, p_n) : p_i \geq 0 \sum p_i = 1\}$$

Σ_n is the fundamental simplex in n-dimensional euclidean space

Back to the previous example

	cu	cv	du	dv
aeg	(10,2)	(10,2)	(0,1)	(0,1)
aeh	(10,2)	(10,2)	(1,2)	(1,2)
afg	(0,2)	(0,2)	(0,1)	(0,1)
afh	(0,2)	(0,2)	(1,2)	(1,2)
beg	(2,3)	(8,2)	(2,3)	(8,2)
beh	(2,3)	(8,2)	(2,3)	(8,2)
bfg	(2,3)	(8,2)	(2,3)	(8,2)
bfh	(2,3)	(8,2)	(2,3)	(8,2)

Observe: the table has pairs repeated several times: different strategies
can lead to the same outcomes.

Revisiting Zermelo

In terms of strategies here is Zermelo's theorem:

Theorem

In the chess game one of the following alternatives holds:

1. *the white has a winning strategy*
2. *the black has a winning strategy*
3. *both players have a strategy leading them at least to a tie*

Outcomes chess in strategic form 1

	1	2	3	...	K
a	...	...	...	...	...
b	...	...	...	...	...
c	W	W	W	...	W
...	...	...	...	...	...
K	...	...	...	...	...

The White has a winning strategy

Outcomes chess in strategic form 2

	1	2	3	...	K
a	...	...	B	...	...
b	...	...	B	...	...
c	...	...	B	...	...
...	...	...	...	...	...
K	...	...	B	...	...

The Black has a winning strategy

Outcomes chess in strategic form 3

	1	2	3	...	K
a	...	...	B	...	...
b	...	...	T	...	...
c	T	W	T	...	W
...	...	...	...	...	...
...	...	...	T	...	...

The outcome of the game is a Tie

This is excluded

T	B	W
W	T	B
B	W	T

This is the case excluded by Zermelo

Remark

If $P_i = \{v_1, \ldots, v_k\}$ and v_j has n_j children then the number of strategies of Player i is $n_1 \cdot n_2 \cdot \cdots \cdot n_k$

This shows that the number of strategies even in short games is usually very high. For instance, if Tic-Tac-Toe is stopped after three moves, the first player has (not exploiting symmetries) $9 \cdot 7^{72}$ strategies.

Games with imperfect information

Sometimes players must make moves at the same time. This can be still represented with a tree.
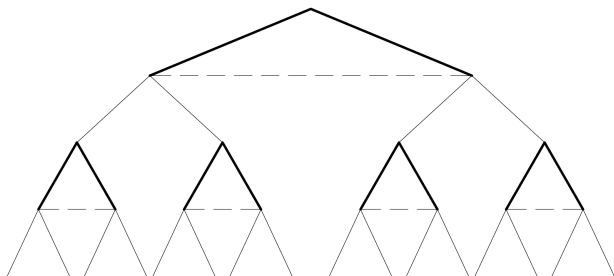

Figure: The prisoner dilemma repeated twice, after one shot the players see the result

Information set

Definition

An information set for a player i is a pair $(U_i, A(U_i))$ with the following properties:

1. *$U_i \subset P_i$ is a nonempty set of vertices $v_1, \ldots, v_k$*
2. *each $v_j \in U_i$ has the same number of children*
3. *$A_i(U_i)$ is a partition of the children of $v_1 \cup \cdots \cup v_k$ with the property that each element of the partition contains exactly one child of each vertex v_j*

Meaning: player i knows to be in U_i, but not in which vertex she is; the partition has the meaning that each set of the partition represents an available move for the player.
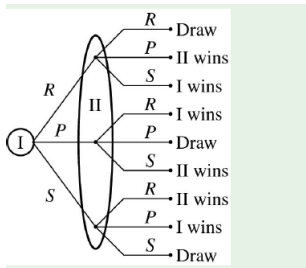
Example

Figure: The Rock-Scissors-Paper game. The information set of player II has three nodes, the number of children of each node is three: thus there are 9 edges. The partition of these edges is made by three elements: A_1 contains RRR, A_2 SSS, A_3 PPP

Picture taken form slides of Prof. Zamir

Formal definition

Definition

An *Extensive form Game with imperfect information* is constituted by

1) A finite set $N = \{1, \ldots, n\}$ of players
2) A game tree (V, E, x_0)
3) A partition made by sets $P_1, P_2, \ldots, P_{n+1}$ of the vertices which are not leaves
4) A partition $(U_i^j), j = 1, \ldots, k_i$ of the set P_i, for all i, with (U_i^j, A_i^j) information set for all i for all j
5) A probability distribution, for each vertex in P_{n+1}, defined on the edges going from the vertex to its children
6) An n-dimensional vector attached to each leaf

Strategies for imperfect information games

Definition

A *pure strategy* for player i in an imperfect information game is a function defined on the collection $\mathcal{U}$ of his information sets and assigning to each U_i in $\mathcal{U}$ an element of the partition $A(U_i)$. A *mixed strategy* is a probability distribution over the pure strategies.

Remark

1. A game of perfect information is a particular game of imperfect information where all information sets of all players are singletons

2. If in a perfect information game $P_i = \{v_1, \ldots, v_k\}$ and each v_i has n_i children, then the number of strategies available to player i is . . .