1. Rationality and consequences

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Summary of the slides

1. Description of the game
2. Players are egoistic
3. Rationality assumptions
4. Preferences
5. Utility functions
6. Probability laws
7. Deepness of analysis
8. Decision theory
9. Dominated strategies
10. First consequences

Setting

Optimization

1. One decision maker
2. At least two decision makers

Possible variants with one decision maker:

1. Scalar optimization
2. Vector optimization
3. Deterministic optimization
4. Stochastic optimization
5. ...

Many possible variants with many decision makers

1. Game theory
2. Social choice
3. Mechanism design
4. Machine learning
5. ...

Crucial difference: the best to do is easily definable when there is one decision maker, much more difficult when many decision makers

Description of the game

A process that can be described by:

1. A set of players (with more than one element)
2. An initial situation
3. The way the players must act and all their available moves
4. All possible final situations
5. The preferences of all agents on the set of the final situations

Examples:

1. The chess game, and all parlour games
2. Two people bargaining how to divide a pie
3. A burglar and a guard
4. Parties in a Parliament
5. Firms competing in a market
6. ...

Games are efficient models for an enormous amount of everyday life situations

Assumptions of the theory

Players are

1. Egoistic
2. Rational

Egoistic means that the player cares only about her own preferences on the outcomes of the game

This is not an ethical issue, but a mathematical assumption, aimed at correctly define what making rational decisions means

Rationality is a much more involved issue

Preferences

Definition

Let X be a set. A binary relation $\succeq$ on a set X is called:

- *reflexive, if for each $x \in X$, xRx*
- *transitive, if for each $x, y, z \in X$, xRy and $yRz \Rightarrow xRz$*
- *total, or complete, if for each $x, y \in X$, $x \neq y \Rightarrow xRy$ or xRy*
- *antisymmetric, if for each $x, y \in X$, xRy and $yRx \Rightarrow x = y$*

A reflexive, transitive and total binary relation on X is called a total preorder (also called, a ranking) on X. A reflexive, transitive, total and antisymmetric binary relation on X is called a total order on X.

The first rationality assumption reads:

The agents are able to provide a preference relation over the outcomes of the game.

Utility functions

Definition

Let $\succeq$ be a preference relation over X. A utility function representing $\succeq$ is a function $u : X \to \mathbb{R}$ such that

$$u(x) \geq u(y) \Longleftrightarrow x \succeq y.$$

1. A utility function need not to exist, however it exists in general setting, in particular if X is a finite set
2. When a utility function exists, then infinite utility functions do exist, since any strictly increasing transformation of a utility function is still a utility function

The second rationality assumption reads:

The agents are able to provide a utility function representing their preferences relations, whenever necessary

Allais experiment 1

First shop

Alternative A

gain	probability
2500	33%
2400	66%
0	1%

Alternative B:

gain	probability
2500	0%
2400	100%
0	0%

In a sample of 72 people exposed to this experiment, 82% of them decided to play the Lottery B.

This is rational if $\frac{34}{100}u(2400) > \frac{33}{100}u(2500)$.

Allais experiment 2

Second shop

Alternative C

gain	probability
2500	33%
0	67%

Alternative D:

gain	probability
2400	34%
0	66%

83% of the people interviewed selected lottery C.

This is rational if $\frac{34}{100}u(2400) < \frac{33}{100}u(2500)$.
Thus it is $A \Leftrightarrow C$: Allais experiment shows that usually agents are not rational players!

Probability issues

The third rationality assumption reads:

The players use consistently the probability laws, in particular they are consistent w.r.t the calculation of expected utilities, they are able to update probabilities according to Bayes rule...

Ability to make a complete analysis

This can be explained with the following example:

Write an integer between 1 and 100

The mean M is calculated

Those writing the number at the minimum distance from qM win the game $(0 < q < 1)$

The player imagined by the theory will answer 1 for every q, with little chance to win.

This necessity to make guesses over the guesses of the guesses is sometimes called The beauty contest, after a remark by the economist Keynes

Deepness of the analysis

The fourth rationality assumption reads:

The players are able to understand consequences of all actions, consequences of this information on any other player, consequences of the consequences...

Extending decision theory

Finally, the fifth rationality assumption reads:

The player are utility maximizers

Given a set of alternatives X, and a utility function u on X, the player seeks an $\bar{x} \in X$ such that

$$u(\bar{x}) \geq u(x), \forall x \in X.$$

In particular, game theory, considering the degenerate case of just one player in the game, extends decision theory.

Summarizing the rationality assumptions

1. The players are able to rank the outcomes of the game
2. The players are able to provide a utility function for their ranking
3. The players apply the expected value principle to built their utility function in presence of random events
4. The players are able to analyze all consequences of their actions, and the consequences of the consequences and so on
5. The players maximize their utilities...

A first concrete consequence of the axioms

A basic consequence of the “decision theory” assumption is:

A player does not take an action a it she has available an action b providing her a strictly better result, no matter what the other players do

Mathematically, let the utility of one player be $u(x, y)$, where $x \in X$ is the space of her possible actions, and y a parameter representing combination of actions of the other players. The assumption says that $a \in X$ cannot be chosen if there is $b \in X$ such that

$$u(b, y) > u(a, y), \forall y$$

Principle of elimination of strictly dominated actions.

Player one action set is $\{18, \dots, 30\}$, player two action set is $\{$accept, refuse$\}$. If player two preference is passing the exam with any grade, rather than repeating it, the action *refuse* is strictly dominated. (Asking for raising the grade by one or two points is not an available action)

An example

Player 1 chooses a row, player 2 a column. The obtained item is a pair, the first (second) digit is the utility of Player 1 (2):

$$\begin{pmatrix} (8,8) & (2,7) \\ (7,2) & (0,0) \end{pmatrix}$$

Utilities of player 1:

$$\begin{pmatrix} 8 & 2 \\ 7 & 0 \end{pmatrix}$$

The second row is strictly dominated by the first, thus player 1 will select the first row

Even if this principle is usually not very informative, it has surprising consequences

Better Argentina or Italy?

Comparisons of games

The first:

$$\begin{pmatrix} (10,10) & (3,15) \\ (15,3) & (5,5) \end{pmatrix}$$

The second one:

$$\begin{pmatrix} (8,8) & (2,7) \\ (7,2) & (0,0) \end{pmatrix}$$

Observe: in any outcome the players are better off in the first game rather than in the second

However it is more convenient for them to play the second!

Less is better than more

The first game:

$$\begin{pmatrix} (10,10) & (3,5) \\ (5,3) & (1,1) \end{pmatrix}$$

The second game, containing all possible outcomes the first, and some further outcomes:

$$\begin{pmatrix} (1,1) & (11,0) & (4,0) \\ (0,11) & (10,10) & (3,5) \\ (0,4) & (5,3) & (1,1) \end{pmatrix}$$

Having less available actions can make the players better off!

Uniqueness issue

$$\begin{pmatrix} (0,0) & (1,1) \\ (1,1) & (0,0) \end{pmatrix}$$

Rational outcomes of this game?

We formally do not know but it is obvious that the rational outcomes will be $(1,1)$

(First row, second column) and (second row, first column) cannot be distinguished and this creates a coordination problem between the players

Elimination of dominated strategies

A votation. Three players, alternatives A, B, C. Players preferences:

$$A \quad \succsim_{\neq 1} {}^{1} \quad B \quad \succsim_{\neq 1} \quad C$$

$$B \quad \succsim_{\neq 2} \quad C \quad \succsim_{\neq 2} \quad A$$

$$C \quad \succsim_{\neq 3} \quad A \quad \succsim_{\neq 3} \quad B$$

In case of three different votes, the alternative selected by player one is winning

What can we expect as rational outcome of the game?

Try with elimination of dominated actions...

[1] $A \succsim_{\neq} B$ means $A \succeq B$ and not $B \succeq A$

The voting game

- Alternative A is a weakly dominant strategy for Player 1
- Players 2 and 3 have as weakly dominated strategy to play their worst choice

Thus the game reduces to

A	A
C	A

whose outcome is C, since both players 2 and 3 like better C than A

However this result is the worst one for the first player!

This shows that eliminating weakly dominated strategies is not always a good idea.