Rationality and consequences

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Setting

Optimization

1) One decision maker
2) At least two decision makers

Possible variants with one decision maker (scalar optimization, vector optimization)

Many possible variants with many decision makers

Game theory, Social choice, Mechanism design

Crucial difference:

The *best* to do is easily definable with one decision maker, much more difficult with many decision makers

Loose Description of game

A process that can be described by:

1) A set of players (with more than one element)
2) An initial situation
3) The way the players must act and all their available moves
4) All possible final situations
5) The preferences of all agents on the set of the final situations

Examples:

1) The chess game
2) Two people bargaining how to divide a pie

Games are efficient models for an enormous amount of everyday life situations.

Assumptions of the theory.

Players are

1) Egoistic
2) Rational

Egoistic means that the player cares only about her own preferences on the outcomes of the game.

Observe this is NOT an ethical issue, but a mathematical assumption.

Rationality is a much more involved issue.

Preferences

Definition

Let X be a set. A preference relation on X is a binary relation $\succeq$ fulfilling, for all $x, y, z \in X$:

1) $x \succeq x$ (*reflexivity*)

2) *either* $x \succeq y$ *or* $y \succeq x$ (*completeness*)

3) $x \succeq y \wedge y \succeq z$ *imply* $x \succeq z$ (*transitivity*)

The first rationality assumption reads:

The agents are able to provide a preference relation over the outcomes of the game.

Utility functions

Definition

Let $\succeq$ be a preference relation over X. A utility function representing $\succeq$ is a function $u : X \to \mathbb{R}$ such that

$$u(x) \geq u(y) \iff x \succeq y.$$

1) A utility function need not to exist, however it exists in general setting, in particular if X is a finite set
2) When a utility function exists, then infinite utility functions do exist

The second rationality assumption reads:

The agents are able to provide a utility function representing their preferences relations, whenever necessary

Allais experiment 1

First shop

Alternative A

gain	probability
2500	33%
2400	66%
0	1%

Alternative B:

gain	probability
2500	0%
2400	100%
0	0%

In a sample of 72 people exposed to this experiment, 82% of them decided to play the Lottery B.

Rational if $\frac{34}{100}u(2400) > \frac{33}{100}u(2500)$).

Allais experiment 2

Second shop
Alternative C

gain	probability
2500	33%
0	67%

Alternative D:

gain	probability
2400	34%
0	66%

83% of the people interviewed selected lottery C.

Rational if $\frac{34}{100}u(2400) < \frac{33}{100}u(2500))$.
Thus $A \Leftrightarrow C$!

Probability issues

The third rationality assumption reads:

The players use consistently the probability laws, in particular they are consistent w.r.t the calculation of expected utilities, they are able to update probabilities according to Bayes rule...

The beauty contest

Another experiment

Write an integer between 1 and 100.

I evaluate the mean M.

The winner(s) is (are) the person(s) writing the number at the minimum distance from $\frac{2}{3}M$.

The player imagined by the theory will answer 1 (with little chance to win).

Deepness of the analysis

The fourth rationality assumption reads:

The players are able to understand consequences of all actions, consequences of this information on any other player, consequences of the consequences...

Extending decision theory

The fifth rationality assumption reads:

The player is able to use decision theory, whenever it is possible.

An obvious consequence is the a player acting alone behaves as forecast by classical decision theory (utility maximizer)

Summarizing the rationality assumptions

1) The players are able to rank the outcomes of the game
2) The players are able to provide a utility function for their ranking
3) The players apply the expected value principle to built their utility function in presence of random events
4) The players are able to analyze all consequences of their actions, and the consequences of the consequences and so on
5) The players use the apparatus of decision theory anytime it is possible

A first concrete consequence of the axioms

A basic consequence of the "decision theory" assumption is:

A player does not take an action *a* it she has available an action *b* providing her a strictly better result, *no matter what the other players do*

Principle of elimination of strictly dominated actions.

Player one action set is $\{18, \dots, 30\}$, player two action set is $\{\text{accept}\,, \text{refuse}\,\}$. If player two preference is passing the exam with any grade, rather than repeating it, the action *refuse* is strictly dominated. Please note: asking for raising the grade by one or two points is not an available action!

An example

Player 1 chooses a row, player 2 a column. The obtained item is a pair, the first (second) digit is the utility of Player 1 (2):

$$\begin{pmatrix} (8,8) & (2,7) \\ (7,2) & (0,0) \end{pmatrix}.$$

Utilities of player 1:

$$\begin{pmatrix} 8 & 2 \\ 7 & 0 \end{pmatrix}.$$

The second row is strictly dominated by the first.

Better Argentina or Italy?

Comparisons of games

The first:

$$\begin{pmatrix} (10,10) & (3,15) \\ (15,3) & (5,5) \end{pmatrix}.$$

The second one:

$$\begin{pmatrix} (8,8) & (2,7) \\ (7,2) & (0,0) \end{pmatrix}.$$

Observe: in any outcome the players are better off in the first game rather than in the second:

However it is more convenient for them to play the second!

Less is better than more

The first game:

$$\begin{pmatrix} (10,10) & (3,5) \\ (5,3) & (1,1) \end{pmatrix}.$$

The second game, containing all possible outcomes the first, and some further outcomes:

$$\begin{pmatrix} (1,1) & (11,0) & (4,0) \\ (0,11) & (10,10) & (3,5) \\ (0,4) & (5,3) & (1,1) \end{pmatrix}.$$

Having less available actions can make the players better off!

Uniqueness issue

$$\begin{pmatrix} (0,0) & (1,1) \\ (1,1) & (0,0) \end{pmatrix}.$$

Rational outcomes of this game?

We do not know but…

(First row, second column) and (second row, first column) cannot be distinguished and this creates a coordination issue.

Elimination of dominated strategies

A votation. Three palyers, alternatives A, B, C. Players preferences:

$$A \quad \succsim_{\neq 1}{}^{1} \quad B \quad \succsim_{\neq 1} \quad C$$

$$B \quad \succsim_{\neq 2} \quad C \quad \succsim_{\neq 2} \quad A$$

$$C \quad \succsim_{\neq 3} \quad A \quad \succsim_{\neq 3} \quad B$$

In case of three different votes, the alternative selected by player one is winning.

What can we expect as rational outcome of the game?

Try with elimination of dominated actions...

[1] $A \succsim_{\neq} B$ means $A \succeq B$ and not $B \succeq A$