Social Choice

Roberto Lucchetti

Politecnico di Milano

Summary of the slides

Social choice

Social Choice:

1. A set A, called the set of the alternatives
2. A set N, called the set of the agents
3. Preferences $\succcurlyeq_i$ of every agent $i \in N$ over the set A of the alternatives

Goal: to find a rule that, to any $(A, N, \succcurlyeq_i)$ associates either a new preference over A, or an alternative in A, i.e. to find a suitable collective choice, given individual preferences

1. Elections
2. Selecting an agenda of works in a condominium
3. Selecting a TV program, in a family during a cold winter evening

$\succcurlyeq = (\succcurlyeq_1, \dots, \succcurlyeq_n)$ is called a preference profile

A votation example

21 voters and three candidates: A, B, and C

Preferences

1	1 voter	A	B	C
2	7 voters	A	C	B
3	7 voters	B	C	A
4	6 voters	C	B	A

Who is elected?

The winner(s)

1. The Condorcet method. Make pairwise comparisons: Candidate C wins, she beats both A and B by 13 to 8
2. The winner is the candidate getting more votes. Candidate A wins, with 8 votes, while B gets 7 votes and C 6 votes
3. Two round process. Those getting the two best ranking have a second contest. Candidates A and B go to the second round, and the winner is B

This is Democracy

Definition 1

Let A be a (finite) set of alternatives.

Definition

A total preorder or a preference relation on A is a subset P of $A \times A$

1. *for all $x \in A$, $(x, x) \in P$ (reflexivity)*
2. *$(x, y) \in P \ \wedge \ (y, z) \in P$ implies $(x, z) \in P$ transitivity*
3. *for every $x, y \in A$ either $(x, y) \in P$ or $(y, x) \in P$ (completeness)*

A total strict preorder on A is a subset P of $A \times A$ fulfilling transitivity, completeness and

1. *For no $x \in A$, $(x, x) \in P$*

$\mathcal{P}$ is the set of all possible preferences over A, $\mathcal{SP}$ is the set of all strict preferences over A

Definition 3

Definition

A social welfare function is a function

$$F : \mathcal{SP}^n \rightarrow \mathcal{P}$$

A social choice function f is a function

$$f : \mathcal{SP}^n \rightarrow A$$

Input
a strict preference profile $(\succ_1, \succ_2, \dots, \succ_n)$

Output

-
- for a social welfare function a preference (i.e. a ranking over the alternatives)
-
- for a social choice function an alternative (i.e. the selected alternative, the first in the list of a preference)

A first method: properties I

When the alternatives are two (and assuming an odd number of voters) Simple majority rule is the rule

Definition

The social welfare function is called anonymous if for every permutation $\pi : N \to N$, if $\pi(\succ_1, \dots, \succ_n)$ denotes the preference profile $(\succ_{\pi(1)}, \dots, \succ_{\pi(n)})$, then

$$F(\pi(\succ_1, \dots, \succ_n)) = F(\succ_1, \dots, \succ_n)$$

In other words, changing the labels to the voters does not change the final result

A first method: properties II

Definition

The social welfare function is called neutral *if for every permutation* $\pi : A \to A$, *if* $\pi(\succ_1, \ldots, \succ_n)$ *denotes the preference profile obtained in the following way*

$$x\pi(\succ_i)y \Leftrightarrow \pi(x) \succ_i \pi(y)$$

then

$$\pi(F(\pi(\succ_1, \ldots, \succ_n))) = F(\succ_1, \ldots, \succ_n)$$

In other words, changing the labels to the alternatives does not change the final result

A first method: properties III

Definition

A social choice function f is *monotonic* *if for every pair of profiles such that*

- $f(\succ) = x$
- $x \succ_i y$ *implies* $x \supset_i y$ *for every* i

then $f(\supset) = x$

In other words, if the alternative x is selected by f according to a given preference profile $\succ$, it will be selected for every other preference profile $\supset$ under which x is, for every agent i, ranked not worse than in $\succ$

Observe, when the alternatives are two, the definition applies also to a social welfare function

The beauty of the simple majority rule

Theorem

if $|A| = 2$ (and there are an odd number of voters), then the simple majority rule is the unique social choice-welfare function fulfilling anonymity, neutrality and monotonicity

What about simple majority if the alternatives are more than two?

Example

$A = \{a, b, c\},\ N = \{1, 2, 3\}.$

1	2	3
a	b	c
b	c	a
c	a	b

As a result of the majority rule

a	b	c	a

The majority rule cannot be extended to more than two alternatives

The Borda method

Giving points to alternatives, according to the ranking

Example

3 voters four alternatives a, b, c, d. Giving 4 points to the first ranked, ..., 1 to the last one

1	2	3
a	d	d
b	c	c
c	b	b
d	a	a

1	2	3
c	d	d
b	c	c
a	b	b
d	a	a

In the first case the result is

d	c	b	a

In the second case the result is

c	d	b	a

Voter 1 is able to change the final ranking between c and d without changing their relative order in her preference

A property

Property

$$[\forall i \quad (x \succ_i y) \Leftrightarrow (x \sqsupset_i y)] \Longrightarrow ((x \succ_N y) \Leftrightarrow (x \sqsupset_N y))$$

Independence from irrelevant alternatives

Another property

Property

$$[\forall i \quad x \succ_i y] \Longrightarrow (x \succ_N y)$$

Unanimity

Last property

Property

there exists $j \in N$ such that

$$F(\succ_1, \dots, \succ_i, \dots, \succ_n) = \succ_j,$$

for each $(\succ_i)_i \in \mathcal{P}^n$.

Existence of a dictator

Note: F is a projection

The Arrow theorem

Theorem

Let A be a set of alternatives such that $|A| \geq 3$. Let N be the set of the agents. Suppose

$$F : \mathcal{SP}^n \to \mathcal{P}$$

is a social welfare function fulfilling:

- *the unanimity property*
- *the irrelevant alternatives property*

Then F admits a dictator

The Arrow theorem:alternative statement

Theorem

let A be a set of alternatives such that $|A| \geq 3$. Let N be the set of the agents. Then there does not exist a social welfare function:

$$F : \mathcal{SP}^n \to \mathcal{P}$$

such that:

- *fulfills the unanimity property*
- *fulfills the irrelevant alternatives property*
- *is not dictatorial*

Proof:step 1

As a first step, it is easy to show that a dictatorial welfare function fulfills IIA and unanimity. Thus the core of the proof is to consider a generic F fulfilling IIA and unanimity, and show that it must be dictatorial. The next of the proof is dedicated to this.

Consider a profile with the following shape, where a is an arbitrary alternative.

1	2	3	...	$n-1$	n
a	a	a	...	...	...
...	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	a	a

We show that for this profile F puts a either at the top or at the bottom of the alternatives: a cannot be in between other alternatives

Continued

Suppose on the contrary there are b, c such that F ranks

...	b	...	a	...	c	...

Consider

1	2	3	...	$n-1$	n
a	a	a	...	c	c
c	c	c	...	...	...
...	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	a	a

1. The mutual positions of a and c and a and b are the same in the two profiles
2. From the IIA property F ranks the same way a, b and a, c in the two profiles
3. Thus in the second one b is ranked before a and a is ranked before c
4. By transitivity in the second one b is ranked before c
5. This contradicts the unanimity property for F

■

Second step

In this second step we single out the player that will be the dictator. To this we fix the following profile

1	2	3	...	$n-1$	n
...	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	...	...
a	a	a	a	a	a

Now agent 1 changes preferences and the new profile is

1	2	3	...	$n-1$	n
a	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	...	...
...	a	a	a	a	a

Since in the first profile f ranks a at the last place by unanimity, now, from the first step of the proof, there are (only) two possibilities

1. F in the second profile ranks a at the first place
2. F in the second profile continues to rank a at the last place

If 1 holds, go to step 3, otherwise consider a new profile where, with respect to the second one, also agent 2 ranks a at the first place. Again, either 1 or 2 must hold. In the first case, go to step 3, otherwise repeat the procedure using now agent 3.

Clearly for some agent 1 must hold, because if the first $n-1$ fail, when agent n will rank a at the first place, then by unanimity also F will rank a at the first place

Third step:1

Suppose the procedure of step 2 selected player j. We now prove that for each alternative b, c such that $b, c \neq a$, for every profile of preferences, F ranks b and c in the same way as j. In this way it is proved that j is "almost" the dictator. Fix an arbitrary profile of the form

1	2	...	j	...	n
...	...	...	...	...	...
...	...	...	b	...	...
...	...	...	...	...	...
...	...	...	c	...	...

We need to show that in this case F ranks alternative b before alternative c. In order to do this, we consider three associated profiles, and we use IIA and transitivity of preferences

Third step:2

1	2	...	j	...	n
...	...	...	...	...	...
...	...	...	b	...	...
...	...	...	...	...	...
...	...	...	c	...	...
...	...	...	...	...	...

1	2	...	j	...	n
a	...	a	...	...	...
...	...	...	b	...	...
...	...	...	...	...	...
...	...	...	c	...	...
...	...	...	a	...	a

1	2	...	j	...	n
a	...	a	a	...	...
...	...	...	b	...	...
...	...	...	...	...	...
...	...	...	c	...	...
...	...	...	...	...	a

1	2	...	j	...	n
a	...	a	...	...	...
...	...	...	b	...	...
...	...	...	a	...	...
...	...	...	c	...	...
...	...	...	...	...	a

Changes of one profile from the previous one are coloured

The following facts hold

- □ Only a is moved in all profiles, thus IIA implies that F ranks b, c in the same order in the four profiles
- ♢ F ranks a at the last place in the second one, in the first place in the third one, due to step 2
- △ mutual positions of a and b are unchanged in the second and fourth profiles, mutual positions of a and c are unchanged in the third and fourth profiles

$\therefore$

By ♢ and △, in the last profile F ranks b better than a and a better than c: by transitivity F ranks b better than c in the last profile. Now the conclusion follows from □: F ranks b better than c in the initial profile

Fourth step: conclusion

We need to show that the agent selected with the procedure in step 2 is able to rank also a with any other alternative. The case $|A| = 3$ can be handled just considering all possible cases.

Consider the case $|A| \geq 4$. It is possible to repeat the complete procedure by selecting another alternative, say b, in steps 1, 2 and 3 of the proof. There are two possibilities

- the agents found when using a and b is the same agent
- the agents found when using a and b are different

In the first case the proof is over. Let us see that the second case leads to a contradiction

Let $a, b, c, d \in A$ and suppose agent 1 is found when fixing a in step 2 and agent 2 is found when fixing b in step 2

Consider any preference profile of the following form

1	2	...
c	d	...
d	c	...
...	...	...
...	...	...

According to agent 1, the welfare function ranks c before d, the opposite according to agent 2, a contradiction ∎

The Social Choice function

Definition

A *social choice function* *f is a function*

$$f : \mathcal{SP}^n \to A$$

Input:= profile of preferences

Output:= an alternative

Properties

Definition

A social choice function f is monotonic if for every pair of profiles such that

- $f(\succ) = x$
- $x \succ_i y$ *implies* $x \sqsupset_i y$

implies $f(\sqsupset) = x$

Definition

A social choice function f is dictatorial if there exists $i \in N$ such that for every profile $\succ$, $f(\succ)$ is the most preferred alternative in $\succ_i$

Definition

A social choice function f satisfies the property of unanimity if every profile $\succ$ such that $a \succ_i b$ for every b and every i, then $f(\succ) = a$

A first theorem

Theorem

If $|A| \geq 3$ a social choice function satisfying unanimity and monotonicity is dictatorial

The Gibbard Satterwhite theorem

Definition

A social choice function f is manipulable if there exist $\succ$, i and $\sqsupset_i$ such that $f(\sqsupset_i, \succ_{-i}) \succ_i f(\succ)$

Theorem

If $|A| \geq 3$ a social choice function satisfying unanimity and non manipulability is dictatorial

Proof of GS theorem

We use the previous theorem and prove that f is monotonic. Suppose on the contrary f is not monotonic. Then there are $\succ, \supset$ and $a \neq b$ such that, for every c

- ▶ $a \succ_i c \quad \Rightarrow a \supset_i c$ for every i
- ♦ $f(\succ) = a \qquad f(\supset) = b$

Fix $a, b \succ, \supset$ fulfilling the above conditions for f, and such that the set of i such that $\succ_i \neq \supset_i$ is minimal (see an example below)

Claim Let $I = \{i : \succ_i \neq \supset_i\}$. Then, if $i \in I$, $f(\succ_i, \supset_{-i}) = a$

Otherwise, there exists $i \in I$ such that $f(\succ_i, \supset_{-i}) = c$, for some $c \neq a$. Then the profile $(\succ_i, \supset_{-i})$ satisfies above conditions with a, c and has less differences with $\succ$ than $\supset$, contradicting minimality

Thus we have the two relations $f(\supset) = b \qquad f(\succ_i, \supset_{-i}) = a$

By non manipulability we get

- □ $b \supset_i a$, since $b = f(\supset) \supset_i f(\succ_i, \supset_{-i}) = a$
- △ $a \succ_i b$, since $a = f((\succ_i, \supset_{-i})) \succ_i f(\supset_i, \supset_{-i}) = f(\supset) = b$

From △ and ▶ we get $a \supset_i b$, contradicting □ ■

An example

$N = \{1, 2, 3, 4\}$, $A = \{a, b, c\}$, as a social choice rule we take f selecting, among the alternatives ranked first by at least one agent, that one coming first in the lexicographic order

1	2	3	4
c	d	d	c
d	b	b	a
b	a	c	d
a	c	a	b

1	2	3	4
c	b	c	c
d	c	d	a
b	d	b	b
a	a	a	a

In the first profile f selects c, in the second one selects b

The same result can be obtained without changing the preference relation of agent 3. Thus here we do not have minimality (to get it, just do not change preference of agent 3)