The Bargaining problem

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Summary of the slides

Setting of the problem

We analyze the problem to modeling the situation of two people bargaining for something.

1. Prototype example: to share a pie (which is seen as 1, so the result will be a percentage of pie each player will get)
2. Observe: it is not a zero sum game, since there is common interest to reach an agreement (otherwise the pie is lost)
3. A non cooperative approach is possible
4. A cooperative approach is possible

Bargaining as extensive game

We start the analysis by considering the ultimatum game (continuous version) Players must divide the quantity 1 between them with the following rules

1. Pl1 proposes division $x = (x_1, x_2)$, x_1 for Pl1 x_2 for Pl1: $x_1 + x_2 = 1$
2. Pl2 either accepts or rejects
3. Outcome x_i for Pli in case of acceptance, 0 for both in case of rejection

Utilities are monetary (risk neutrality)

By backward induction, there is a unique solution: Pl1 proposes $(1, 0)$, Pl2 accepts every offer

What if Pl2 can make a counteroffer?

Two stages

1. At first stage Pl1 proposes (x_1, x_2), then Pl2 either accepts or rejects
2. Acceptance ends the game. Rejection implies replication of the one stage game, with roles interchanged, i.e a counteroffer (y_1, y_2) by Pl2 and acceptance or rejection of Pl1

The subtree following rejection at the first stage by Pl2 is ultimatum game with Players interchanged

Thus the unique outcome by backward induction is $(0, 1)$

This can be easily extended to any number of stages: the last one having the possibility to make an offer gets everything

Impatient players

Suppose Pli has a discount factor $0 < \delta_i < 1$ at each stage

Suppose a two stage deadline

1. At first stage the offer is (x_1, x_2)
2. if accepted, game over, if rejected, at the second stage the offer is (y_1, y_2), with utilities $(\delta_1 y_1, \delta_2 y_2)$

The rest unchanged

Unique backward induction outcome

1. Pl1 offers $(1 - \delta_2, \delta_2)$
2. Pl2 accepts the offer

Why so?

Strategies

1. After any rejection by Pl2, the game becomes ultimatum game with Pl2 starting the game, thus her offer after rejection is always $(0, 1)$, and her utility is δ_2
2. Thus Pl2 accepts an offer x_2 at the first stage if and only if $x_2 \geq \delta_2$
3. Pl1 knows he will get nothing offering less than δ_2
4. Optimal proposal for Pl1 $(1 - \delta_2, \delta_2)$

Strategies of the players

1. Player 1: Proposal of $(1 - \delta_2, \delta_2)$ at the first node, say yes at every node in the second stage
2. Player 2:Accept any offer (x_1, x_2) if and only if $x_2 \geq \delta_2$; otherwise reject the offer and propose $(0, 1)$

Game with infinite horizon

No bound on the number of stages

Possible plays, where $x^k = (x_1^k, x_2^k)$

1. $(x^1, N, x^2, N, \dots, x^n, N, \dots)$ No offer is accepted
2. $(x^1, N, \dots, x^T, Y)$ Offer x^T accepted at (some) time T

Utilities

1. $(0, 0)$
2. $(\delta_1^{T-1} x_1^T, \delta_2^{T-1} x_2^T)$

Subgame perfect equilibrium

Backward does not apply: we need a more general concept, reducing to backward induction in the finite case

Definition

A subgame perfect NEp is a NEp such that its restriction to every subgame of the initial game represents a NEp of the subgame

If the game is finite, a perfect equilibrium profile is what is obtained by applying backward induction

The structure of the game

These facts are obvious

1. At every stage the same game is played, in alternate stages the roles of the players are interchanged
2. An offer of (x_1, x_2) at the first stage produces the same game situation as the offer (x_1, x_2) at stage $2k + 1$, with the same preferences of the players: only the discount factor applies

Looking for special strategies

The structure of the game suggests that the strategy of the players should be of the form:
Propose a certain division w and accept an offer z if and only in z the player gets at least a fixed quota:

1. Pl1 proposes $\bar{x}$ and accepts y if and only if $y_1 \geq \bar{y}_1$
2. Pl2 proposes $\bar{z}$ and accepts w if and only if $w_2 \geq \bar{w}_2$

for suitable parameters $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{w}$

Looking at the two stage game it is clear that

1. $\bar{w}_2$ represents the minimum level of acceptance for Pl2. Thus an offer $x_2 < w_2$ forces a rejection
2. optimality for Player one implies $\bar{x}_2 = \bar{w}_2$
3. the same argument applied to the second player provides $\bar{z}_1 = \bar{y}_1$

$\therefore$

$\bar{x} = \bar{w}$ and $\bar{z} = \bar{y}$

Relating $\bar{x}$ and $\bar{y}$

Thus

1. Pl1 proposes $\bar{x}$ and accepts y if and only if $y_1 \geq \bar{y}_1$
2. Pl2 proposes $\bar{y}$ and accepts x if and only if $x_2 \geq \bar{y}_2$

How to relate $\bar{x}$ and $\bar{y}$? Suppose Player 1 makes the offer $\bar{x} = (\bar{x}_1, \bar{x}_2)$, knowing that Player 2 strategy is to offer $\bar{y} = (\bar{y}_1, \bar{y}_2)$. If Player 2 will offer this next stage, this means that he wants utility $\delta_2 \bar{y}_2$ and will reject any lower offer. Thus the best offer of Player 1 that will be accepted by Player 2 is

$$\bar{x}_2 = \delta_2 \bar{y}_2$$

The situation is symmetric for Player 2.

The strategies

The above considerations lead to the following conjecture:

$$\boxed{\bar{x}_2 = \delta_2 \bar{y}_2, \qquad \bar{y}_1 = \delta_1 \bar{x}_1}$$

Since $\bar{x}_2 = 1 - \bar{x}_1$ and $\bar{y}_2 = 1 - \bar{y}_1$

$$\bar{x} = (\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2})$$

$$\bar{y} = (\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}, \frac{(1-\delta_1)}{1-\delta_1\delta_2})$$

The result

Theorem

There is a unique subgame perfect equilibrium for the bargaining game with alternate offers and impatient players, and the following are the strategies

1. *Pl1: if he must make a proposal, this is $\bar{x}$; if he has to either accept or reject a proposal y, he accepts it if and only if $y_1 \geq \bar{y}_1$*
2. *Pl2 : if he must make a proposal, this is $\bar{y}$; if he has to either accept or reject a proposal x, she accepts it if and only if $x_2 \geq \bar{x}_2$*

where

$$\bar{x} = (\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2})$$

$$\bar{y} = (\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}, \frac{(1-\delta_1)}{1-\delta_1\delta_2})$$

The outcome of the game

1. Pl1 offers $\bar{x}$ to Pl2
2. Pl2 accepts the offer at the first stage

Utilities

1. Player 1
$$\frac{1-\delta_2}{1-\delta_1\delta_2}$$
2. Player 2
$$\frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$$

The game ends at the first stage

Proof partial

Call σ_i the strategy of Pli. Let us fix any subtree. We want to show that in the subtree the restriction of (σ_1, σ_2) still is a N.E.profile.

We need to prove that no deviation of one player, fixed the strategy of the other one, provides a better result to the deviating player.

We call τ another strategy profile, and consider the first node v where σ differs from τ.

Two cases can occur

1. At the node v the player must propose
2. At the node v the player must either accept or reject a proposal

An offer must be done

in the first case, suppose Player 1 is considering a deviation. With the NEp (σ_1, σ_2) his proposal is $x = (x_1, x_2)$, and gets payoff $\bar{x}_1$. Suppose he offers something different from $\bar{x}_2$:

1. Suppose he offers more than x_2. Since Pl2 uses strategy σ_2, he accepts the offer, thus Pl1 gets less than $\bar{x}_1$. Thus offering something greater than x_2 is not optimal for Pl1
2. Suppose he offers less than x_2. In this case Pl2 rejects the offer and proposes $\bar{y}_1$ to Pl1. If Pl1 accepts the offer, according to σ_2 he gets $\bar{y}_1 = \delta_1 \bar{x}_1 < \bar{x}_1$. Since he would get less than playing σ_1, he must refuse the offer. It is then his turn to make an offer. Now again he can offer something more than x_2, but this is not convenient as just seen, or he can offer x_2, and the offer is accepted, but his utility is smaller, because of the discount factor δ_1, or he can offer less and the argument repeats.

This shows that it is not convenient for Player 1 to deviate in a node where she is called to make an offer

Starting with a response

Suppose now the subgame starts with Pl2 giving an answer to a proposal made by Pl 1 at node v. Suppose this offer is x_2. The strategy $\bar{\sigma}_2$ specifies that Pl2 accepts the offer x if and only if $x_2 \geq \bar{x}_2$. Let us see if for her it is convenient to deviate from σ_2 and taking for granted that Pl 1 plays σ_1 (since we compare two stages of the game we "forget" the fact that we are at a stage T and we actualize utilities as if the stages were stage 1 and stage two: if you do not like it, all utilities should be multiplied by δ_2^{T-1})

What happens if Player 2 plays σ_2

1. Case $x_2 < \bar{x}_2$. Player 2 refuses the offer, counteroffers $\bar{y}_1$, his offer is accepted. Payoff for Pl2, $\bar{y}_2$ with utility $\delta_2\bar{y}_2 > x_2$
2. Case $x_2 > \bar{x}_2$. Player 2 accepts the offer, and his utility is x_2

what happens if layer 2 deviates from σ_2:

1. Case $x_2 < \bar{x}_2$. Player 2 accepts the offer, with utility $x_2 < \delta_2\bar{y}_2$
2. Case $x_2 > \bar{x}_2$. Player 2 refuses the offer and makes a counteroffer. The best offer to do is $\bar{y}_1$ (according to σ_1 any offer less than $\bar{y}_1$ is refused), the proposal is accepted, the payoff is $\bar{y}_2$ with utility $\delta_2\bar{y}_2 = \bar{x}_2 < x_2$

In both cases for the player is not convenient to deviate. To conclude we observe that the two players are symmetric, thus the above considerations hold for both.

Uniqueness is much more tricky...

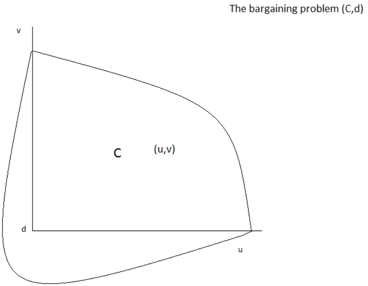
The symmetric case

When $\delta_1 = \delta_2 := \delta$, the final utilities of the players are

$$(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$$

showing, as expected, that in case of symmetric players the first to talk has an advantage

Definition of bargaining problem, according to Nash



1. d is the disagreement point: d_i is the utility of player i if an agreement is not reached
2. C is the set of all possible (utility) outcomes: $(u, v) \in C$ means that a possible outcome of the bargaining process assigns utility u (v) to player 1 (2)
3. This is as a cooperative game (NTU) with two players

The set of the bargaining problems

$\mathcal{C} = \{(C, d)\}$ such that

1. C is closed bounded convex subset of $\mathbb{R}^2$
2. $d \in \mathbb{R}^2$
3. there exists $x \in C : x_1 > d_1, x_2 > d_2$

1. C closed bounded is no restrictive assumption
2. Convexity is more delicate but acceptable
3. Assumption on x means that both players have interest in bargaining

The solution concept

Definition

A solution for the bargaining problem is a function

$$f : \mathcal{C} \to \mathbb{R}^2$$

such that $f[(C,d)] \in C$, *for all* $(C,d) \in \mathcal{C}$

Properties for a solution

The following are interesting properties for f:

1. Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is the following transformation of the plane: $L(x_1, x_2) = (ax_1 + c, bx_2 + e)$, with $a, b > 0$ and $c, e \in \mathbb{R}$. Then
$$f[L(C), L(d)] = L[f(C, d)]$$
2. Suppose $S : \mathbb{R}^2 \to \mathbb{R}^2$ is the following transformation of the plane: $S(x_1, x_2) = (x_2, x_1)$. Suppose moreover a game (C, d) fulfills $(S(C), S(d)) = (C, d)$. Then
$$f(C, d) = S[f(C, d)]$$
3. Given the two problems (A, d) and (C, d)if
$$A \supset C \wedge f[(A, d)] \in C$$
then $f[(C, d)] = f[(A, d)]$
4. $y \in C \wedge u \in C : u_1 > y_1, u_2 > y_2$
implies $f[(C, x)] \neq y$

Meaning

The properties are called

1. Invariance with respect to admissible transformations of utility functions
2. Symmetry. In a problem (C, d) fulfilling $(S(C), S(d)) = (C, d)$ the players are symmetric
3. Independence from irrelevant alternatives, for short IIA
4. Efficiency

Remark

The function L providing admissible transformation of utility functions is invertible: $L^{-1}(y_1, y_2) = (\frac{y_1}{a} - \frac{c}{a}, \frac{y_2}{b} - \frac{d}{b})$ represents an admissible transformation of utility functions as well

The Nash bargaining theorem

Theorem

There is one and only one f satisfying the above properties. Precisely, if $(C, d) \in \mathcal{C}$, $f[(C, d)]$ is the point maximizing the function

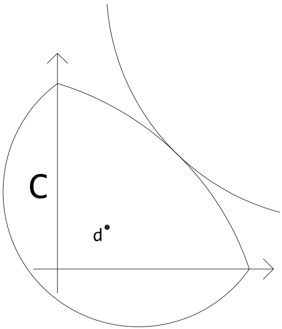
$$g(u, v) = (u - d_1)(v - d_2)$$

on the set

$$C \cap \{(u, v) : u \geq d_1, v \geq d_2\}$$

In other words, players must maximize the product of their utilities over the set of the interesting outcomes

The solution graphically



The proof

Proof Outline.

1. f is well defined: the point maximizing g on C exists, since g is a continuous function and the domain C is closed convex bounded. Uniqueness of the maximum point is provided by strict quasi concavity of the function g.
2. The verification that C satisfies the other properties is not difficult. In particular IIA is trivial, and efficiency is straightforward
3. Uniqueness: call h a function fulfilling the properties. Symmetry and efficiency imply $h = f$ on the subclass of the symmetric games. Now take a general problem (C, d) and, by means of the property of invariance with respect to admissible transformation of utilities send d to the origin and the point $f(C, d)$ to $(1, 1)$
   $(L(x_1, x_2) = (\frac{1}{f_1[(C,d)]-d_1}x_1 - \frac{d_1}{f_1[(C,d)]-d_1}, \frac{1}{f_2[(C,d)]-d_2}x_2 - \frac{d_2}{f_2[(C,d)]-d_2})$. Then
   $$L(C) \subset A = \{(u, v) : \ u, v \geq 0, \ u + v \leq 2\}$$
   $(A, 0)$ is a symmetric game, so that $f(A, 0) = h(A, 0) = (1, 1)$. The independence of irrelevant alternatives provides $h(L(C), 0) = (1, 1) = f(L(C), 0)$. Apply again the property of invariance with respect to admissible transformation of utilities to go back to the original bargaining situation, and conclude from this.

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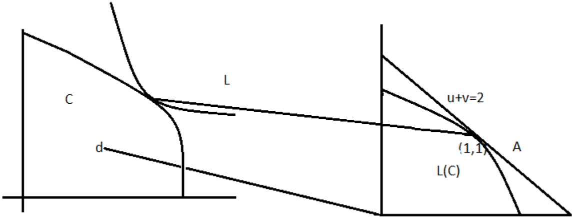
Picture for uniqueness



The transformation L sends d to $(0,0)$ and the Nash solution to $(1,1)$. Apply IIA to $(L(C),0)$ and $(A,(0,0)$ to conclude that $h[(L(C),(0,0)] = f[(L(C),(0,0)]$ and go back with the inverse of L

An interesting fact (1)

The problem is dividing a pie of 1, Player one will get x Player 2 $1-x$ with utilities $u_1(x), u_2(1-x)$ with u_i is increasing, concave and twice differentiable such that $u_i(0)=0$

x must maximize $g(z)=u_1(z)u_2(1-z)$

It must be $g'(x)=0$. Thus the equation:

$$\frac{u_1'(x)}{u_1(x)}=\frac{u_2'(1-x)}{u_2(1-x)}$$

must hold

The two curves $\frac{u_1'(z)}{u_1(z)}$ and $\frac{u_2'(1-z)}{u_2(1-z)}$ intersect at the unique point with abscissa x

An interesting fact (2)

Suppose the second player changes his utility function from u_2 to $h \circ u_2$, h as u_i, call y the new quantity assigned to Player 1

The above equation becomes:

$$\frac{u_1'(y)}{u_1(y)} = \frac{h'(u_2(1-y))u_2'(1-y)}{h(u_2(1-y))}$$

Since for every z

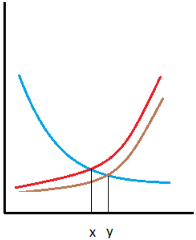
$$\frac{u_2'(1-z)}{u_2(1-z)} \geq \frac{h'(u_2(1-z))u_2'(1-z)}{h(u_2(1-z))}$$

it follows $y > x$

Applying h to u_2 means that the second player becomes more risk averse

Thus according to Nash the more risk averse one player is, the less he get: a well known fact in experimental economics.

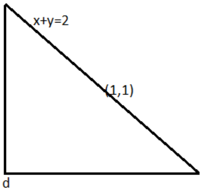
An interesting fact: the picture

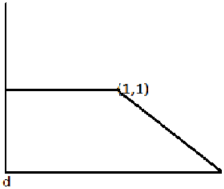


$$\frac{u_1'(x)}{u_1(x)} \quad \frac{u_2'(1-x)}{u_2(1-x)} \quad \frac{h'(u_2(1-x))u_2'(1-x)}{h(u_2(1-x))}$$

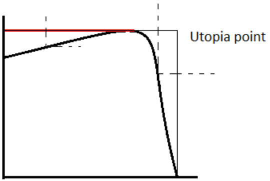
How realistic is the model?

1. The least realistic assumption: player's utilities are common knowledge
2. Convexity is a bit restrictive
3. Uniqueness is based on the fact that the domain of the function is quite large
4. The IIA assumption can be criticized





Alternative assumption



$$g_C(x) = \begin{cases} y & \text{if } (x,y) + \mathbb{R}^2_+ \cap C = (x,y) \\ U_2 & \text{otherwise} \end{cases}$$

$U = (U_1, U_2) :=$ Utopia point, where $U_i = \max u_i$ on $C \cap \{(u_1, u_2) : u_1 \geq d_1, u_2 \geq d_2\}$

Monotonicity assumption

Definition

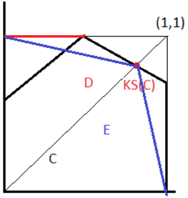
Let $f : \mathcal{C} \to \mathbb{R}^2$ be a solution of the bargaining problem. Then f satisfies the monotonicity assumption for player 2 if for every pair of problems (C, d), $(\hat{C}, d)$ such that $U_1[(C, d)] = U_1[(\hat{C}, d)]$ and $g_C \leq g_{\hat{C}}$, it holds that $f_2[(\hat{C}, d)] \geq f_2[(C, d)]$

Theorem

There is one and only one solution f fulfilling efficiency, invariance with respect to admissible transformation of utilities, symmetry and monotonicity for both players: f associates to every (C, d) the efficient point lying on the line joining the points d and U

f is called the Kalai-Smorodinski solution

Proof of the KS theorem



Let f be any function fulfilling the axioms. In the picture

1. All problems have $(0,0)$ as disagreement point, and $(1,1)$ as utopia point
2. The problem E is symmetric
3. In every symmetric problem KS and f must coincide: $f(E) = KS(E)$
4. by monotonicity $f(E) = f(D) = f(C)$, $KS(E) = KS(D) = KS(C)$

$\therefore$

$f(C) = KS(C)$. By invariance with respect to admissible transformation of utilities it is $f[(C,d)] = KS[(C,d)]$ for all (C,d) ■