Bargaining with alternate offers

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Bargaining as extensive game

Ultimatum game (continuous version) Players must divide the quantity 1 between them

- Pl1 proposes division $x = (x_1, x_2)$, x_1 for Pl1 x_2 for Pl1: $x_1 + x_2 = 1$
- Pl2 either accepts or rejects
- Outcome x_i for Pli in case of acceptance, 0 for both in case of rejection

Utilities are monetary (risk neutrality)

By backward induction: Pl1 proposes $(1, 0)$, Pl2 accepts every offer

Unique solution

What about NEp?

What if Pl2 can make a counteroffer?

Two stages

- At first stage Pl1 proposes (x_1, x_2), then Pl2 either accepts or rejects
- Acceptance ends the game. Rejection implies replication of the one stage game, with roles interchanged, i.e a counteroffer (y_1, y_2) by Pl2 and acceptance or rejection of Pl1

The subtree following rejection at the first stage by Pl2 is ultimatum game with Players interchanged

Thus the unique outcome by backward induction is $(0, 1)$

Strategies?

This can be easily extended to any number of stages

Impatient players

Suppose Pli has a discount factor $0 < \delta_i < 1$ at each stage

Suppose a two stage deadline

- At first stage the offer is (x_1, x_2)
- if accepted, game over, if rejected, at the second stage the offer is (y_1, y_2), with utilities $(\delta_1 y_1, \delta_2 y_2)$

The rest unchanged

Unique backward induction outcome

1) Pl1 offers $(1 - \delta_2, \delta_2)$
2) Pl2 accepts the offer

Strategies

- After any rejection by Pl2, the game becomes ultimatum game with Pl2 starting the game, thus her offer after rejection is alwaysis $(0, 1)$, with utility δ_2
- Thus Pl2 accepts an offer x_2 at the first stage if and only if $x_2 \geq \delta_2$
- Pl1 knows he will get nothing offering less than δ_2
- Optimal proposal for Pl1 $(1 - \delta_2, \delta_2)$

Strategies of the players

- Player 1: Proposal of $(1 - \delta_2, \delta_2)$ at the first node, say yes at every other node
- Accept any offer (x_1, x_2) if and only if $x_2 \geq \delta_2$; otherwise reject the offer and propose $(0, 1)$

Game with infinite horizon

No bound on the number of stages

Possible plays

- $(x^1, N, x^2, N, \dots)$ No offer is accepted
- $(x^1, N, \dots, x^T, Y)$ Offer x^T accepted at time T

Utilities

- $(0, 0)$
- $(\delta_1^{T-1} x_1^T, \delta_2^{T-1} x_2^T)$

Subgame perfect equilibrium

Backward induction cannot be applied: need of a more general concept, reducing to backward induction in the finite case

Definition

A subgame perfect NEp is a NEp such that its restriction to every subgame of the initial game represents a NEp of the subgame

If the game is finite, a perfect equilibrium profile is what is obtained by applying backward induction

The structure of the game

These facts are obvious

- At every stage the same game is played, in alternate stages the roles of the players are interchanged
- An offer of (x_1, x_2) at the beginning produces the same game as the offer (x_1, x_2) at stage $2k + 1$, with the same preferences of the players: only the discount factor applies

Looking for special strategies

The structure of the game suggest that the strategy of the players should be of the form: a proposal of a certain division and acceptance of any offer if and only if the offer overcomes some selected quota. Thus

- Pl1 proposes $\bar{x}$ and accepts y if and only if $y_1 \geq \bar{y}_1$
- Pl2 proposes $\bar{z}$ and accepts w if and only if $w_2 \geq \bar{w}_2$

for suitable parameters $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{w}$

- $\bar{w}_2$ represents the minimum level of acceptance for Pl2. Thus an offer $x_2 < w_2$ forces a rejection
- optimality implies $\bar{x}_2 = \bar{w}_2$
- by symmetry $\bar{z}_1 = \bar{y}_1$

$\therefore$

$\bar{x} = \bar{w}$ and $\bar{z} = \bar{y}$

Relating $\bar{x}$ and $\bar{y}$

Thus

- Pl1 proposes $\bar{x}$ and accepts y if and only if $y_1 \geq \bar{y}_1$
- Pl2 proposes $\bar{y}$ and accepts x if and only if $x_2 \geq \bar{y}_2$

How to relate $\bar{x}$ and $\bar{y}$? A good conjecture, according to the two stage case, is

$$\bar{x}_2 = \delta_2 \bar{y}_2, \qquad \bar{y}_1 = \delta_1 \bar{x}_1$$

Since $\bar{x}_2 = 1 - \bar{x}_1$ and $\bar{y}_2 = 1 - \bar{y}_1$

$$\bar{x} = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$$

$$\bar{y} = \left(\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}, \frac{(1-\delta_1)}{1-\delta_1\delta_2}\right)$$

The result

Theorem

There is a unique subgame perfect equilibrium for the bargaining game with alternate offers and impatient players, and the following are the strategies

1) *Pl1: if he must make a proposal, this is $\bar{x}$; if he has to either accept or reject a proposal y, he accepts it if and only if $y_1 \geq \bar{y}_1$*
2) *Pl2 : if he must make a proposal, this is $\bar{y}$; if he has to either accept or reject a proposal x, she accepts it if and only if $x_2 \geq \bar{x}_2$*

where

$$\bar{x} = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$$

$$\bar{y} = \left(\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}, \frac{(1-\delta_1)}{1-\delta_1\delta_2}\right)$$

The outcome of the game

- Pl1 offers $\bar{x}$ to Pl2
- Pl2 accepts the offer at the first stage

Utilities

- Player 1

$$\frac{1-\delta_2}{1-\delta_1\delta_2}$$

- Player 2

$$\frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$$

The game ends at the first stage

Proof partial

Call σ_i the strategy of Pli. There are two possible cases

1) A subgame starting with an offer
2) A subgame starting with a response to an offer

We need to prove that in any case the strategy profile $\sigma = (\sigma_1, \sigma_2)$, restricted to the subgame, is a NEp

Starting with an offer

Suppose the subgame starts at the node v where Pl1 must make an offer. With the NEp (σ_1, σ_2) he gets payoff $\bar{x}_1$. Suppose PLl1 offers something different from $\bar{x}_2$:

- Suppose he offers more than x_2. Since Pl2 accepts the offer, Pl1 gets less than $\bar{x}_1$. Thus offering something greater than x_2 is not optimal
- Suppose he offers less than x_2. In this case Pl2 rejects the offer and proposes $\bar{y}_1$ to Pl1. If Pl1 accepts the offer, the counteroffer is $\bar{y}_1 = \delta_1 \bar{x}_1 < \bar{x}_1$. Since he would get less than playing σ_1, he must refuse the offer. Now again he can offer something more than x_2, but this is not convenient as just seen, or he can again offer less. But in this case again Pl2 refuses and the situation repeats again...

We need now to show that also for Pl2 is not convenient to deviate when the subgame starts at the node v. Thus Pl 2 faces the situation of answering to an offer

Starting with a response

Suppose now the subgame starts with Pl2 giving an answer to a proposal made by Pl 1 at node v. Suppose this offer is x_2.
The strategy $\bar{\sigma}_2$ specifies that Pl2 accepts the offer x if and only if $x_2 \geq \bar{x}_2$. Let us see if for her it is convenient to deviate

- Case $x_2 < \bar{x}_2$. Since σ_2 here specifies a rejection, deviating means accepting the offer. In this case the payoff is x_2. With σ_2 the counteroffer $\bar{y}_1$: her offer is accepted (according to σ_1) with payoff, for Pl2 of $\bar{y}_2$ and utility $\delta_2 \bar{y}_2 > x_2$: thus for her deviating form σ_2 is not profitable
- Case $x_2 > \bar{x}_2$. A deviation means refusing the offer. But in this case the optimal proposal is $\bar{y}_1$ (according to σ_1 any offer less than $\bar{y}_1$ is refused), the proposal will be accepted, the payoff is $\bar{y}_2$ with utility $\delta_2 \bar{y}_2 = \bar{x}_2 < x_2$: thus for her deviating form σ_2 is not profitable

Uniqueness is much more tricky...

The symmetric case

When $\delta_1 = \delta_2 := \delta$, the final utilities of the players are

$$(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$$

showing, as expected, that in case of symmetric players the first to talk has an advantage