Zero-Sum Games

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Zero sum games

An interesting case of non cooperative game is is when there are two players, with opposite interests.

Definition

A two player zero sum game in strategic form is given by strategy sets X and Y and a payoff function $f : X \times Y \to \mathbb{R}$

Conventionally $f(x, y)$ is what Player I gets from Player II, when they play x, y respectively. The payoff for Player II is $-f(x, y)$ (she pays $f(x, y)$ to Player I).

Finite zero-sum games

In the finite case $X = \{1, 2, \ldots, n\}$, $Y = \{1, 2, \ldots, m\}$ the game is described by a payoff matrix P

Example

$$P = \begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}$$

Player I selects row i, Player II selects column j.

Conservative values

$$\begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}$$

Player I can guarantee herself to get at least

$$v_1 = \max_i \min_j p_{ij} = \max\{1, 5, 0\} = 5$$

Player II can guarantee himself to pay no more than

$$v_2 = \min_j \max_i p_{ij} = \min\{8, 5, 8\} = 5$$

Rational outcome: $i^* = 2$, $j^* = 2$, Value $v_1 = v_2 = 5$

Alternative idea of solution

Suppose

- $v_1 = v_2 := v$
- i^* is the row attaining the $\max_i \min_j p_{ij} = v$ so that $p_{i^*j} \geq v$ for all j
- j^* is the column attaining the $\min_j \max_i p_{ij} = v$ so that $p_{ij^*} \leq v$ for all i

Then $p_{i^*j^*} = v$ is the rational outcome of the game.

Remark

- *i^* is an optimal strategy for Player I, because he cannot get more than v, since v is the conservative value of Player II*
- *j^* is an optimal strategy for Player II, because he cannot pay less than v, since v is the conservative value of Player I*

The equality $v_1 = v_2$ need not hold

Example

$$P = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$v_1 = -1$, $v_2 = 1$

Nothing unexpected...

General zero-sum games

$f : X \times Y \to \mathbb{R}$

The players can guarantee to themselves (almost) the conservative values:

Player I: $v_1 = \sup_x \inf_y f(x, y)$

Player II: $v_2 = \inf_y \sup_x f(x, y)$

We always have $v_1 \leq v_2$

$v_1 \leq v_2$

Proposition

$$v_1 = \sup_x \inf_y f(x,y) \leq \inf_y \sup_x f(x,y) = v_2$$

Proof Observe that, for all x, y,

$$\inf_y f(x,y) \leq f(x,y) \leq \sup_x f(x,y)$$

Thus

$$\inf_y f(x,y) \leq \sup_x f(x,y)$$

Since the left hand side does not depend on y and the right hand side does not depend on x, the thesis follows. ■

Saddle points and Nash equilibria of zero-sum games

Suppose that we have strategies $\bar{x}$ and $\bar{y}$ such that for all $y \in Y$ and $x \in X$

$$f(x, \bar{y}) \leq f(\bar{x}, \bar{y}) \leq f(\bar{x}, y).$$

Such a pair $(\bar{x}, \bar{y})$ is called a *saddle point*.

Let $v = f(\bar{x}, \bar{y})$. Then

- The rational outcome of the game is $v_1 = v_2 = v$
- $\bar{x}$ is an optimal strategy for Player I
- $\bar{y}$ is an optimal strategy for Player II

The Nash equilibria of a zero sum game

Theorem

Let X, Y be nonempty strategy sets and $f : X \times Y \to \mathbb{R}$. Then the following are equivalent:

1. *The pair $(\bar{x}, \bar{y})$ is a Nash equilibrium, i.e. fulfills*

$$f(x, \bar{y}) \leq f(\bar{x}, \bar{y}) \leq f(\bar{x}, y) \quad \forall x \in X, \, \forall y \in Y$$

2. *The following conditions are satisfied:*
(i) $\inf_y \sup_x f(x, y) = \sup_x \inf_y f(x, y)$*: the two conservative values agree*
(ii) $\inf_y f(\bar{x}, y) = \sup_x \inf_y f(x, y)$*: $\bar{x}$ is optimal for Player I*
(iii) $\sup_x f(x, \bar{y}) = \inf_y \sup_x f(x, y)$*: $\bar{y}$ is optimal for Player II*

Proof

Proof 1) implies 2). From 1) we get:

$$\inf_y \sup_x f(x,y) \leq \sup_x f(x,\bar{y}) = f(\bar{x},\bar{y}) = \inf_y f(\bar{x},y) \leq \sup_x \inf_y f(x,y)$$

Since $v_1 \leq v_2$ always holds, all above inequalities are equalities

Conversely, suppose 2) holds Then

$$\inf_y \sup_x f(x,y) \stackrel{(iii)}{=} \sup_x f(x,\bar{y}) \geq f(\bar{x},\bar{y}) \geq \inf_y f(\bar{x},y) \stackrel{(ii)}{=} \sup_x \inf_y f(x,y)$$

Because of (i), all inequalities are equalities and the proof is complete ∎

Mixed extension of a zero-sum game

Zero-sum finite game: $n \times m$ matrix P.

Mixed strategy space for Player I:

$$X = \Sigma_n = \{x = (x_1, \ldots, x_n) : x_i \geq 0, \sum_{i=1}^{n} x_i = 1\}$$

Mixed strategy space for Player II:

$$Y = \Sigma_m = \{y = (y_1, \ldots, y_m) : y_j \geq 0, \sum_{j=1}^{m} y_j = 1\}$$

Expected payoff:

$$f(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij} x_i y_j = x^t P y$$

To prove existence of a rational outcome

What must be proved to have existence of a rational outcome:

1. $v_1 = v_2$
2. there exists $\bar{x}$ such that

$$v_1 = \sup_x \inf_y f(x,y) = \inf_y f(\bar{x},y)$$

3. there exists $\bar{y}$ such that

$$v_2 = \inf_y \sup_x f(x,y) = \sup_x f(x,\bar{y})$$

In the finite case $\bar{x}$ and $\bar{y}$ fulfilling 2) and 3) always exist; thus it suffices to establish 1).

The von Neumann theorem

Theorem

A two player, finite, zero sum game as described by a payoff matrix P has a rational outcome: the two conservative values of the players coincide and there are optimal strategies $\bar{x}$, $\bar{y}$ for the players.

Remark

We remind that when the two conservative values agree the strategy $\bar{x}$ is optimal for Player I if and only if it guarantees her to get the (common conservative) value no matter what Player II does; dually the strategy $\bar{y}$ is optimal for Player II if and only if it guarantees him to get the (common conservative) value no matter what Player I does.

Finding optimal strategies: Player I

Player I must choose a probability distribution $\Sigma_n \ni x = (x_1, \ldots, x_n)$:

$$
\begin{array}{l}
p_{11}x_1 + \cdots + p_{n1}x_n \geq v \\
\ldots \\
p_{1j}x_1 + \cdots + p_{nj}x_n \geq v \\
\ldots \\
p_{1m}x_1 + \cdots + p_{nm}x_n \geq v
\end{array}
$$

where v must be as large as possible

$$
(P_1) \left\{ \begin{array}{l}
\max v \\
P^t x \geq v 1_m \\
1^t x = 1 \\
x \geq 0, v \in \mathbb{R}
\end{array} \right.
$$

Finding optimal strategies: Player II

Player II must choose a probability distribution $\Sigma_m \ni y = (y_1, \ldots, y_m)$:

$$
\begin{aligned}
&p_{11}y_1 + \cdots + p_{1m}y_m \leq w \\
&\ldots \\
&p_{i1}y_1 + \cdots + p_{im}y_m \leq w \\
&\ldots \\
&p_{n1}y_1 + \cdots + p_{nm}y_m \leq w
\end{aligned}
$$

where w must be as small as possible

$$
(P_2)\begin{cases} \min w \\ Py \leq w1_n \\ 1^t y = 1 \\ y \geq 0, w \in \mathbb{R} \end{cases}
$$

In matrix form

$$(P_1)\begin{cases}\max v\\ P^t x \geq v1_m\\ 1^t x = 1\\ x \geq 0, v \in \mathbb{R}\end{cases} \qquad (P_2)\begin{cases}\min w\\ Py \leq w1_n\\ 1^t y = 1\\ y \geq 0, w \in \mathbb{R}\end{cases}$$

These linear programs are dual to each other !

Both are feasible $\Rightarrow$ they have optimal solutions and there is no duality gap $v = w$

The complementarity conditions

The complementarity conditions become

- $x_i > 0 \Longrightarrow \sum_{j=1}^{m} p_{ij} y_j = v$
- $y_j > 0 \Longrightarrow \sum_{i=1}^{n} p_{ji} x_i = w$

Since $\sum_{i=1}^{n} p_{ji} x_i$ is the expected value for Player II if she plays column j and Player I the mixed strategy $x = (x_1, \ldots, x_n)$, the complementarity conditions show, one more time, that a Player must give a positive probability only to those pure strategies that have optimal expected payoff.

Summarizing

A finite zero sum game has always rational outcome in mixed strategies.

The set of Nash equilibria can be found by solving a pair of dual linear programming problems.

The outcome, at each pair of optimal strategies, is the common conservative value v of the players.

The set of optimal strategies for the players is a nonempty closed convex set, the smallest convex set containing a finite number of points, called the extreme points of the set.

As a consequence of the theorem

- Every Nash equilibrium $(\bar{x}, \bar{y})$ of the zero sum game provides optimal strategies for the players
- Any pair of optimal strategies for the players provides a Nash equilibrium for the zero sum game

Thus Nash theorem is a generalization of von Neumann theorem

A comment

Remark

Von Neumann approach with conservatives values shows that, in the particular case of the zero sum game:

- *Each player can find an optimal strategy independently of the other player.*
- *Any pair of optimal strategies provides a Nash equilibrium; this implies no need of coordination to reach an equilibrium.*
- *Every Nash equilibrium provides the same utility (payoff) to the players: multiplicity of solutions does not create problems.*
- *Nash equilibria are easy to be found in zero sum games.*

Fair games

Definition

A square matrix $n \times n$ $P = (p_{ij})$ is said to be anti-symmetric provided $p_{ij} = -p_{ji}$ for all $i, j = 1, \ldots, n$. A finite zero sum game is said to be fair if the associated matrix is antisymmetric.

Example: Rock-Scissors-Paper.

In fair games there is no favorite player.

Fair outcome

Proposition

In a fair game

- *the value is* 0
- $\bar{x}$ *is an optimal strategy for Player I if and only if it is optimal for Player II*

Proof Since

$$x^t P x = (x^t P x)^t = x^t P^t x = -x^t P x,$$

$f(x,x) = 0$ for all x, thus $v_1 \leq 0, v_2 \geq 0$

Then $v = 0$.

If $\bar{x}$ is optimal for the Player I, $\bar{x}^t P y \geq 0$ for all y

Thus $y^t P \bar{x} \leq 0$ for all $y \in \Sigma_n$, and $\bar{x}$ is optimal for Player II ∎

Finding optimal strategies in a fair game

Need to solve the system of inequalities

$$
\begin{aligned}
&p_{11}x_1 + \cdots + p_{n1}x_n \geq 0 \\
&\ldots \\
&p_{1j}x_1 + \cdots + p_{nj}x_n \geq 0 \\
&\ldots \\
&p_{1m}x_1 + \cdots + p_{nm}x_n \geq 0
\end{aligned}
$$

with the extra conditions:

$$
x_i \geq 0, \qquad \sum_{i=1}^{n} x_i = 1
$$

A proposed exercise

Exercise

Find the optimal strategies of the players in the rock,scissors, paper game and in the following fair game:

$$P = \begin{pmatrix} 0 & 3 & -2 & 0 \\ -3 & 0 & 0 & 4 \\ 2 & 0 & 0 & -3 \\ 0 & -4 & 3 & 0 \end{pmatrix}$$