Zero-Sum Games

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Contents of the week

- Zero sum games
- Conservative values
- Von Neumann theorem
- Fair games
Zero sum games

An interesting case of non cooperative game is when there are two players, with opposite interests.

**Definition**

A *two player zero sum game* in strategic form is given by strategy sets $X$ and $Y$ and a payoff function $f : X \times Y \rightarrow \mathbb{R}$

Conventionally $f(x, y)$ is what Player I gets from Player II, when they play $x$, $y$ respectively. The payoff for Player II is $-f(x, y)$ (she pays $f(x, y)$ to Player I).
Finite zero-sum games

In the finite case $X = \{1, 2, \ldots, n\}$, $Y = \{1, 2, \ldots, m\}$ the game is described by a payoff matrix $P$

Example

$$P = \begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}$$

Player I selects row $i$, Player II selects column $j$. 
Conservative values

\[
\begin{pmatrix}
4 & 3 & 1 \\
7 & 5 & 8 \\
8 & 2 & 0
\end{pmatrix}
\]

Player I can guarantee herself to get at least

\[v_1 = \max_i \min_j p_{ij} = \max\{1, 5, 0\} = 5\]

Player II can guarantee himself to pay no more than

\[v_2 = \min_j \max_i p_{ij} = \min\{8, 5, 8\} = 5\]

Rational outcome: \(i^* = 2, j^* = 2\), Value \(v_1 = v_2 = 5\)
Alternative idea of solution

Suppose

- \( \nu_1 = \nu_2 :\nu \)
- \( i^* \) is the row attaining the \( \max_i \min_j p_{ij} = \nu \) so that \( p_{i^*j} \geq \nu \) for all \( j \)
- \( j^* \) is the column attaining the \( \min_j \max_i p_{ij} = \nu \) so that \( p_{ij^*} \leq \nu \) for all \( i \)

Then \( p_{i^*j^*} = \nu \) is the rational outcome of the game.

Remark

- \( i^* \) is an optimal strategy for Player I, because he cannot get more than \( \nu \), since \( \nu \) is the conservative value of Player II
- \( j^* \) is an optimal strategy for Player II, because he cannot pay less than \( \nu \), since \( \nu \) is the conservative value of Player I
The equality $v_1 = v_2$ need not hold

Example

$P = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

$v_1 = -1, \ v_2 = 1$

Nothing unexpected...
General zero-sum games

\[ f : X \times Y \rightarrow \mathbb{R} \]

The players can guarantee to themselves (almost) the conservative values:

Player I: \( v_1 = \sup_x \inf_y f(x, y) \)

Player II: \( v_2 = \inf_y \sup_x f(x, y) \)

We always have \( v_1 \leq v_2 \)
Proposition

\[ v_1 = \sup_x \inf_y f(x, y) \leq \inf_y \sup_x f(x, y) = v_2 \]

Proof

Observe that, for all \( x, y \),

\[ \inf_y f(x, y) \leq f(x, y) \leq \sup_x f(x, y) \]

Thus

\[ \inf_y f(x, y) \leq \sup_x f(x, y) \]

Since the left hand side does not depend on \( y \) and the right hand side does not depend on \( x \), the thesis follows.
Saddle points and Nash equilibria of zero-sum games

Suppose that we have strategies $\bar{x}$ and $\bar{y}$ such that for all $y \in Y$ and $x \in X$

$$f(x, y) \leq f(\bar{x}, \bar{y}) \leq f(\bar{x}, y).$$

Such a pair $(\bar{x}, \bar{y})$ is called a saddle point.

Let $\nu = f(\bar{x}, \bar{y})$. Then

- The rational outcome of the game is $\nu_1 = \nu_2 = \nu$
- $\bar{x}$ is an optimal strategy for Player I
- $\bar{y}$ is an optimal strategy for Player II
The Nash equilibria of a zero sum game

Theorem

Let $X$, $Y$ be nonempty strategy sets and $f : X \times Y \to \mathbb{R}$. Then the following are equivalent:

1. The pair $(\bar{x}, \bar{y})$ is a Nash equilibrium, i.e. fulfills

   $$ f(x, \bar{y}) \leq f(\bar{x}, \bar{y}) \leq f(\bar{x}, y) \quad \forall x \in X, \forall y \in Y $$

2. The following conditions are satisfied:
   (i) $\inf_y \sup_x f(x, y) = \sup_x \inf_y f(x, y)$: the two conservative values agree
   (ii) $\inf_y f(\bar{x}, y) = \sup_x \inf_y f(x, y)$: $\bar{x}$ is optimal for Player I
   (iii) $\sup_x f(x, \bar{y}) = \inf_y \sup_x f(x, y)$: $\bar{y}$ is optimal for Player II
Proof

1) implies 2). From 1) we get:

\[
\inf_y \sup_x f(x, y) \leq \sup_x f(x, \bar{y}) = f(\bar{x}, \bar{y}) = \inf_y f(\bar{x}, y) \leq \sup_x \inf_y f(x, y)
\]

Since \(v_1 \leq v_2\) always holds, all above inequalities are equalities.

Conversely, suppose 2) holds. Then

\[
\inf_y \sup_x f(x, y) \overset{(iii)}{=} \sup_x f(x, \bar{y}) \geq f(\bar{x}, \bar{y}) \geq \inf_y f(\bar{x}, y) \overset{(ii)}{=} \sup_x \inf_y f(x, y)
\]

Because of (i), all inequalities are equalities and the proof is complete \(\blacksquare\)
Mixed extension of a zero-sum game

Zero-sum finite game: \( n \times m \) matrix \( P \).

Mixed strategy space for Player I:

\[
X = \Sigma_n = \{ x = (x_1, \ldots, x_n) : x_i \geq 0, \sum_{i=1}^{n} x_i = 1 \}
\]

Mixed strategy space for Player II:

\[
Y = \Sigma_m = \{ y = (y_1, \ldots, y_m) : y_j \geq 0, \sum_{j=1}^{m} y_j = 1 \}
\]

Expected payoff:

\[
f(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij} x_i y_j = x^t P y
\]
To prove existence of a rational outcome

What must be proved to have existence of a rational outcome:

1. \( \nu_1 = \nu_2 \)
2. there exists \( \bar{x} \) such that
   \[
   \nu_1 = \sup_x \inf_y f(x, y) = \inf_y f(\bar{x}, y)
   \]
3. there exists \( \bar{y} \) such that
   \[
   \nu_2 = \inf_y \sup_x f(x, y) = \sup_x f(x, \bar{y})
   \]

In the finite case \( \bar{x} \) and \( \bar{y} \) fulfilling 2) and 3) always exist; thus it suffices to establish 1).
The von Neumann theorem

Theorem

A two player, finite, zero sum game as described by a payoff matrix $P$ has a rational outcome: the two conservative values of the players coincide and there are optimal strategies $\bar{x}$, $\bar{y}$ for the players.

Remark

We remind that when the two conservative values agree the strategy $\bar{x}$ is optimal for Player I if and only if it guarantees her to get the (common conservative) value no matter what Player II does; dually the strategy $\bar{y}$ is optimal for Player II if and only if it guarantees him to get the (common conservative) value no matter what Player I does.
Finding optimal strategies: Player I

Player I must choose a probability distribution $\Sigma_n \ni x = (x_1, \ldots, x_n)$:

\[
p_{11}x_1 + \cdots + p_{n1}x_n \geq v \\
\vdots \\
p_{1j}x_1 + \cdots + p_{nj}x_n \geq v \\
\vdots \\
p_{1m}x_1 + \cdots + p_{nm}x_n \geq v
\]

where $v$ must be as large as possible

\[
(P_1) \begin{cases}
\max v \\
P^t x \geq v1_m \\
1^t x = 1 \\
x \geq 0, v \in \mathbb{R}
\end{cases}
\]
Finding optimal strategies: Player II

Player II must choose a probability distribution $\Sigma \ni y = (y_1, \ldots, y_m)$:

$$p_{11}y_1 + \cdots + p_{1m}y_m \leq w$$
$$\vdots$$
$$p_{i1}y_1 + \cdots + p_{im}y_m \leq w$$
$$\vdots$$
$$p_{n1}y_1 + \cdots + p_{nm}y_m \leq w$$

where $w$ must be as small as possible

$$\begin{cases}
\min w \\
Py \leq w1_n \\
1^t y = 1 \\
y \geq 0, w \in \mathbb{R}
\end{cases}$$

\( (P_2) \)
In matrix form

\[(P_1)\left\{ \begin{array}{l} \max v \\ P^t x \geq v 1_m \\ 1^t x = 1 \\ x \geq 0, \; v \in \mathbb{R} \end{array} \right. \]

\[(P_2)\left\{ \begin{array}{l} \min w \\ Py \leq w 1_n \\ 1^t y = 1 \\ y \geq 0, \; w \in \mathbb{R} \end{array} \right. \]

These linear programs are dual to each other!

Both are feasible $\Rightarrow$ they have optimal solutions and there is no duality gap $v = w$
The complementarity conditions

The complementarity conditions become

- \( x_i > 0 \Rightarrow \sum_{j=1}^{m} p_{ij} y_j = v \)
- \( y_j > 0 \Rightarrow \sum_{i=1}^{n} p_{ji} x_i = w \)

Since \( \sum_{i=1}^{n} p_{ji} x_i \) is the expected value for Player II if she plays column \( j \) and Player I the mixed strategy \( x = (x_1, \ldots, x_n) \), the complementarity conditions show, one more time, that a Player must give a positive probability only to those pure strategies that have optimal expected payoff.
Summarizing

A finite zero sum game has always rational outcome in mixed strategies.

The set of Nash equilibria can be found by solving a pair of dual linear programming problems.

The outcome, at each pair of optimal strategies, is the common conservative value $v$ of the players.

The set of optimal strategies for the players is a nonempty closed convex set, the smallest convex set containing a finite number of points, called the extreme points of the set.
As a consequence of the theorem

- Every Nash equilibrium \((\bar{x}, \bar{y})\) of the zero sum game provides optimal strategies for the players.
- Any pair of optimal strategies for the players provides a Nash equilibrium for the zero sum game.

Thus Nash theorem is a generalization of von Neumann theorem.
A comment

Remark

Von Neumann approach with conservatives values shows that, in the particular case of the zero sum game:

- Each player can find an optimal strategy *independently* of the other player.
- Any pair of optimal strategies provides a Nash equilibrium; this implies no need of coordination to reach an equilibrium.
- Every Nash equilibrium provides the same utility (payoff) to the players: multiplicity of solutions does not create problems.
- Nash equilibria are *easy to be found* in zero sum games.
Fair games

Definition

A square matrix $n \times n \ P = (p_{ij})$ is said to be anti-symmetric provided $p_{ij} = -p_{ji}$ for all $i, j = 1, \ldots, n$. A finite zero sum game is said to be fair if the associated matrix is antisymmetric.

Example: Rock-Scissors-Paper.

In fair games there is no favorite player.
How to find optimal strategies

Fair outcome

Proposition

In a fair game

- the value is 0
- \( \bar{x} \) is an optimal strategy for Player I if and only if it is optimal for Player II

Proof

Since

\[
x^tPx = (x^tPx)^t = x^tP^tx = -x^tPx,
\]

\( f(x, x) = 0 \) for all \( x \), thus \( v_1 \leq 0, v_2 \geq 0 \)

Then \( v = 0 \).

If \( \bar{x} \) is optimal for the Player I, \( \bar{x}^tPy \geq 0 \) for all \( y \)

Thus \( y^tP\bar{x} \leq 0 \) for all \( y \in \Sigma_n \), and \( \bar{x} \) is optimal for Player II \( \square \)
How to find optimal strategies

Finding optimal strategies in a fair game

Need to solve the system of inequalities

\[ p_{11}x_1 + \cdots + p_{n1}x_n \geq 0 \]
\[ \vdots \]
\[ p_{1j}x_1 + \cdots + p_{nj}x_n \geq 0 \]
\[ \vdots \]
\[ p_{1m}x_1 + \cdots + p_{nm}x_n \geq 0 \]

with the extra conditions:

\[ x_i \geq 0, \quad \sum_{i=1}^{n} x_i = 1 \]
A proposed exercise

Exercise

*Find the optimal strategies of the players in the rock, scissors, paper game and in the following fair game:*

\[
P = \begin{pmatrix}
0 & 3 & -2 & 0 \\
-3 & 0 & 0 & 4 \\
2 & 0 & 0 & -3 \\
0 & -4 & 3 & 0
\end{pmatrix}
\]