Existence of Equilibria for Strategic Games

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Topics

- Mixed extension of finite games
- Best response maps
- Nash existence theorem
- Examples of *n*-person games
  - Hotelling game
  - Cournot competition
  - Braess paradox
  - El Farol bar
  - Auctions

Nash equilibrium in pure strategies might not exist

Consider the game

$$\begin{pmatrix} (4,0) & (3,1) \\ (3,5) & (5,0) \end{pmatrix}$$

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There is no equilibrium... in pure strategies.

A player cannot use the same strategy all the time; this would be observed and the other player could take advantage from this.

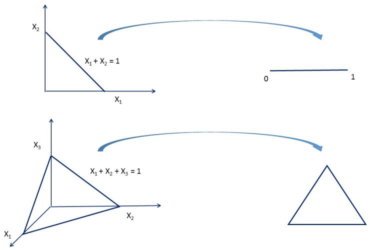
- It makes sense to play strategies according to some probability scheme.
- But these probabilities must be chosen strategically!

Simplexes and mixed strategies

Definition

Let A be a finite strategy set with d elements (also called *pure strategies*). The set of *mixed strategies* over the set A is the d-dimensional simplex

$$\Sigma_A = \{(x_a)_{a\in A} \in \mathbb{R}^A : x_a \geq 0, \textstyle\sum_{a\in A} x_a = 1\}.$$



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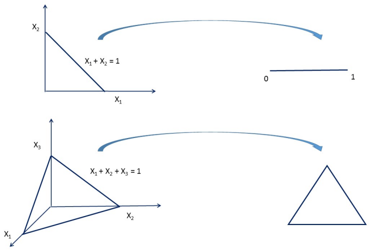

A vector $x = (x_a)_{a\in A} \in \Sigma_A$ defines a probability distribution on the set A with

$$x_a = \mathbb{P}(\text{playing the pure strategy } a)$$

Mixed extension of 2-player games

Consider a 2-person game with strategy sets $I = \{1, \ldots, n\}$ and $J = \{1, \ldots, m\}$, and payoff matrices (A, B). In the *mixed extension* of the game player 1 choses a probability distribution $x \in \Sigma_I$ and player 2 a probability distribution $y \in \Sigma_J$.

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The probability of observing the outcome ij is the product $x_i y_j$ and then the *expected* payoffs for both players are respectively:

Player 1: $f(x,y) = \sum_{i=1}^n \sum_{j=1}^m A_{ij} x_i y_j = x' A y$

Player 2: $g(x,y) = \sum_{i=1}^n \sum_{j=1}^m B_{ij} x_i y_j = x' B y$

Mixed extension of 2-player games

Alternatively:

- The expected payoff for player 1 when playing the pure strategy $i \in I$ against the mixed strategy $y \in \Sigma_J$ of player 2 is

  $$u_i(y) = \sum_{j=1}^{m} A_{ij} y_j$$

  and then

  $$f(x, y) = \sum_{i=1}^{n} x_i u_i(y)$$

- The expected payoff for player 2 when playing the pure strategy $j \in J$ against against the mixed strategy $x \in \Sigma_I$ of player 1 is

  $$v_j(x) = \sum_{i=1}^{n} B_{ij} x_i$$

  and then

  $$g(x, y) = \sum_{j=1}^{m} y_j v_j(x)$$

Best responses

Nash equilibrium relies on the assumption that players maximize their payoff with respect to their own variable, taking for granted the choice of the other player.

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Player 1: $\max_{x\in\Sigma_I} \sum_{i=1}^{n} x_i u_i(y) \quad \Rightarrow \quad BR_1(y) = \operatorname*{Argmax}_{x\in\Sigma_I} \sum_{i=1}^{n} x_i u_i(y)$

Player 2: $\max_{y\in\Sigma_J} \sum_{j=1}^{m} y_j v_j(x) \quad \Rightarrow \quad BR_2(x) = \operatorname*{Argmax}_{y\in\Sigma_J} \sum_{j=1}^{m} y_j v_j(x)$

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Player 2: $\max\limits_{y\in\Sigma_J} \sum_{j=1}^m y_j v_j(x) \quad \Rightarrow \quad BR_2(x) = \operatorname*{Argmax}_{y\in\Sigma_J} \sum_{j=1}^m y_j v_j(x)$

$BR_1(y)$ is the set of all x's that maximize $f(\cdot, y)$ for a fixed $y \in \Sigma_J$.
$BR_2(x)$ is the set of all y's that maximize $g(x, \cdot)$ for a fixed $x \in \Sigma_I$.

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Hence, a Nash equilibrium in mixed strategies is a pair $(\bar{x}, \bar{y}) \in \Sigma_I \times \Sigma_J$ such that

$$\begin{cases} \bar{x} \in BR_1(\bar{y}) \\ \bar{y} \in BR_2(\bar{x}) \end{cases}$$

Example: finding Nash equilibria

Consider the game

$$\begin{pmatrix} (4,0) & (3,1) \\ (3,5) & (5,0) \end{pmatrix}$$

Player 1 selects *Top* or *Bottom* with probabilities $(p, 1-p)$.
Player 2 chooses *Left* or *Right* with probabilities $(q, 1-q)$.

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Player 1 selects *Top* or *Bottom* with probabilities $(p, 1-p)$.
Player 2 chooses *Left* or *Right* with probabilities $(q, 1-q)$.

Then their expected payoffs are respectively

$$\begin{aligned} f(p,q) &= p\, u_T(q) + (1-p)\, u_B(q) \\ &= p\,[4q + 3(1-q)] + (1-p)\,[3q + 5(1-q)] \end{aligned}$$

$$\begin{aligned} g(p,q) &= q\, v_L(p) + (1-q)\, v_R(p) \\ &= q\,[5(1-p)] + (1-q)\,[1p] \end{aligned}$$

Example: finding Nash equilibria

$$BR_1(q) = \underset{p\in[0,1]}{\text{Argmax}}\ \ p\ u_T(q) + (1-p)\ u_B(q)$$

$$u_T(q) = 4q + 3(1-q) \qquad ; \qquad u_B(q) = 3q + 5(1-q)$$

- $q > \frac{2}{3} \Rightarrow u_T(q) > u_B(q) \Rightarrow$ it is optimal for player 1 to choose $p = 1$.
- $q < \frac{2}{3} \Rightarrow u_T(q) < u_B(q) \Rightarrow$ it is optimal for player 1 to choose $p = 0$.
- $q = \frac{2}{3} \Rightarrow u_T(q) = u_B(q) \Rightarrow$ every $p \in [0,1]$ is equally good for player 1.

Example: finding Nash equilibria

$$BR_2(p) = \operatorname*{Argmax}_{q\in[0,1]} \; q\, v_L(p) + (1-q)\, v_R(p)$$

$$v_L(p) = 5(1-p) \qquad ; \qquad v_R(p) = p$$

- $p < \frac{5}{6} \Rightarrow v_L(p) > v_R(p) \Rightarrow$ it is optimal for player 2 to choose $q = 1$.
- $p > \frac{5}{6} \Rightarrow v_L(p) < v_R(p) \Rightarrow$ it is optimal for player 2 to choose $q = 0$.
- $p = \frac{5}{6} \Rightarrow v_L(p) = v_R(p) \Rightarrow$ every $q \in [0,1]$ is equally good for player 2.

Example: finding Nash equilibria

$$BR_1(q) = \begin{cases} \{1\} & \text{if } q > \frac{2}{3} \\ \{0\} & \text{if } q < \frac{2}{3} \\ [0,1] & \text{if } q = \frac{2}{3} \end{cases}$$

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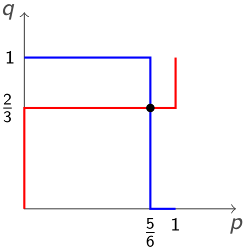
Unique Nash equilibrium: $\bar{p} = \frac{5}{6}$ and $\bar{q} = \frac{2}{3}$



General strategic games

Consider an n-player game with strategy sets X_i and payoffs $f_i : X \to \mathbb{R}$ where as usual $X = \prod_{j=1}^{n} X_j$ is the set of strategy profiles.

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Define the best response map $BR_i : X_{-i} \to X_i$ as

$$BR_i(x_{-i}) = \{x_i \in X_i : f_i(x_i, x_{-i}) \geq f(z_i, x_{-i}) \, \forall z_i \in X_i\}$$

which associates to each possible strategies x_{-i} of the other players the set of x_i's that maximize the payoff $f(\cdot, x_{-i})$.

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which associates to each possible strategies x_{-i} of the other players the set of x_i's that maximize the payoff $f(\cdot, x_{-i})$.

Then: $(\bar{x}_i)_{i=1}^{n}$ is a Nash equilibrium if and only if for each player $i = 1, \ldots, n$ we have $\bar{x}_i \in BR_i(\bar{x}_{-i})$.

The Nash theorem

Theorem

Given a n-player game with strategy sets X_i and payoff functions $f_i : X \to \mathbb{R}$ where $X = \prod_{i=1}^{n} X_i$. Suppose:

- *each X_i is a closed bounded convex subset in a finite dimensional space $\mathbb{R}^{d_i}$*
- *each $f_i : X \to \mathbb{R}$ is continuous*
- *$x_i \mapsto f_i(x_i, x_{-i})$ is a (quasi) concave function for each fixed $x_{-i} \in X_{-i}$*

Then there exists at least one Nash equilibrium.

A simple 2-player game

Let $X = Y = [0, 10]$ and let the payoffs of the players be

$$\begin{aligned} f(x, y) &= -x^2 - 2xy + 12x + 1 \\ g(x, y) &= -y^2 + 2xy + 8y + 7 \end{aligned}$$

The common strategy set is a closed interval, thus a closed convex bounded set. The functions are continuous, and concave in one variable when the other is fixed.

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We get $BR_1(y) = \{\max(0, 6 - y)\}$ and $BR_2(x) = \{\min(10, 4 + x)\}$.
To find an equilibrium we solve

$$\begin{cases} x = \max(0, 6 - y) \\ y = \min(10, 4 + x) \end{cases}$$

with unique solution $(\bar{x}, \bar{y}) = (1, 5)$.

Mixed equilibria for 2-player finite games

Corollary

Every 2-player finite game has at least one Nash equilibrium in mixed strategies.

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In this case

- $\Sigma_I \subseteq \mathbb{R}^I$ and $\Sigma_J \subseteq \mathbb{R}^J$ are simplexes, hence closed bounded and convex
- $f(x,y) = x'Ay$ and $g(x,y) = x'By$ are jointly continuous w.r.t (x,y)
- $f(x,y) = \sum_{i=1}^{n} x_i u_i(y)$ is linear with respect to x (for fixed y) and $g(x,y) = \sum_{j=1}^{m} y_i v_j(x)$ is linear with respect to y (for fixed x)

and thus the assumptions of Nash theorem are fulfilled.

Mixed equilibria for n-player finite games

Consider an n-person finite game with strategy sets A_i and payoffs $f_i(a_1, \ldots, a_n)$. In the *mixed extension* each player i choses a probability distribution $x^i \in \Sigma_{A_i}$, that is to say, $x^i_{a_i} \geq 0$ for all $a_i \in A_i$ and $\sum_{a_i \in A_i} x^i_{a_i} = 1$.

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Denote $A = \prod_{i=1}^n A_i$ the set of pure strategy profiles. The probability of observing an outcome $(a_1, \dots, a_n) \in A$ is the product $\prod_{i=1}^n x^i_{a_i}$ and the *expected* payoffs are:

$$\bar{f}_i(x^1, \dots, x^n) = \sum_{(a_1,\dots,a_n)\in A} f_i(a_1, \dots, a_n) \prod_{j=1}^n x^j_{a_j} = \sum_{a_i \in A_i} x^i_{a_i} u_i(a_i, x^{-i})$$

$$u_i(a_i, x^{-i}) = \sum_{a_j \in A_j, j \neq i} f_i(a_1, \dots, a_n) \prod_{j \neq i} x^j_{a_j}$$

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Corollary

Every n-player finite game has at least one Nash equilibrium in mixed strategies.

Maximizing over a simplex

Consider the problem of maximizing a weighted average

$$
\begin{aligned}
&\text{maximize} \;\; 15x_1 + 18x_2 + 23x_3 + 23x_4 \\
&\text{subject to:} \\
&x_1 + x_2 + x_3 + x_4 = 1 \\
&x_1, \ldots, x_4 \geq 0
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Clearly the optimal value is 23 and is attained by putting all the weight on the variables x_3 and x_4 which have larger coefficients.

In particular $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ are optimal, as well as $(0, 0, \frac{1}{2}, \frac{1}{2})$.

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In general the optimal solutions are all vectors (x_1, x_2, x_3, x_4) with $x_1 = x_2 = 0$ and $x_3, x_4 \geq 0$ with $x_3 + x_4 = 1$.

Maximizing over a simplex

Consider a player with a finite set A of pure strategies. Let $u_a = u_a(x_{-i})$ be the expected payoff when playing action $a \in A$ given that the other player(s) use mixed strategies x_{-i}. Then the best response requires to solve

$$\max_{x \in \Sigma_A} \sum_{a \in A} x_a u_a$$

In order to render the weighted average $\sum_{a \in A} x_a u_a$ as large as possible one simply has to put all the weigth on the variables with largest coefficients u_a.

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Thus, setting $v = \max_{a \in A} u_a$ we actually have

$$\max_{x \in \Sigma_A} \sum_{a \in A} x_a u_a = v$$

and $x \in \Sigma_A$ is a best response (optimal solution) if and only if

$$(\forall a \in A)\ x_a > 0 \Rightarrow u_a = v.$$

Note that there is always a best response in pure strategies: choose any a with maximal u_a and set $x_a = 1$ and $x_{a'} = 0$ for $a' \neq a$.

Maximizing over a simplex

Definition

For a non-negative vector $(x_1, \dots, x_n) \geq 0$ we define its support as the set of indexes of its strictly positive entries, that is

$$spt(x) = \{i : x_i > 0\}$$

Mixed equilibria in 2-player finite games

In a 2-player finite game, the pair $(\bar{x}, \bar{y}) \in \Sigma_I \times \Sigma_J$ is a Nash equilibrium in mixed strategies iff there exists $v, w \in \mathbb{R}$ such that

$$\begin{cases} u_i(\bar{y}) = v & \text{for all } i \in spt(\bar{x}) \\ u_i(\bar{y}) \leq v & \text{for all } i \notin spt(\bar{x}) \end{cases}$$

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In words: Player 1 should play with positive probability only the rows with maximal expected payoff $u_i(\bar{y}) = v$, and symetrically Player 2 should assign positive probability only to the columns with maximal payoff $v_j(\bar{x}) = w$.

- v = equilibrium payoff for Player 1
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$$\begin{cases} \sum_{j=1}^{m} A_{ij}\bar{y}_j = v & \text{for all } i \in spt(\bar{x}) \\ \sum_{j=1}^{m} A_{ij}\bar{y}_j \leq v & \text{for all } i \notin spt(\bar{x}) \end{cases}$$

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"Linear" system of equations and inequalities in $n + m + 2$ unknowns: $\bar{x}_i, \bar{y}_j, v, w$

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$$\begin{cases} \sum_{i=1}^{n} x_i = 1 & \\ \sum_{j=1}^{m} A_{ij} y_j = v & \text{for all } i \in spt(\bar{x}) \\ x_i = 0 & \text{for all } i \notin spt(\bar{x}) \end{cases}$$

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3. Check whether the ignored inequalities are satisfied.
   If $x_i \geq 0, y_j \geq 0, \sum_{j=1}^{m} A_{ij} y_j \leq v$ and $\sum_{i=1}^{n} B_{ij} x_i \leq w$ then Stop: we have found a mixed equilibrium. Otherwise, go back to step 1 and try another guess of the supports.

Exercises

1. Find all the equilibria for the Battle of the sexes

$$\begin{pmatrix} (3,2) & (1,1) \\ (0,0) & (2,3) \end{pmatrix}$$

2. Find all the equilibria for
   - Hawks & Doves
   - Crossing game
   - Prisoner's Dilemma
   - Tragedy of the Commons
   - Rock-Scissors-Paper

Lemke-Howson Algorithm

Enumerating all the possible supports in the brute force algorithm quickly becomes computationally prohibitive: there are potentially $(2^n - 1)(2^m - 1)$ options !

For $n \times n$ games the number of combinations grow very quickly

n	# of potential supports
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Lemke-Howson proposed a more efficient algorithm... though still with exponential running time in the worst case.

Exercise: Search Lemke-Howson's algorithm on the web, download it, and use it to solve the previous games.

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4. With six or more carts there are infinitely many equilibria.

Cournot competition

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Therefore, the utilities of both firms are respectively

$$\begin{aligned} u_1(q_1, q_2) &= q_1\, p(q_1 + q_2) - c\, q_1 \\ u_2(q_1, q_2) &= q_2\, p(q_1 + q_2) - c\, q_2 \end{aligned}$$

The monopolist

Suppose that Firm 2 is temporarily out of business, so that $q_2 = 0$. Then Firm 1 becomes a monopolist and maximizes its utility

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Clearly it makes no sense to produce more than a, so the optimal production lies in the interval $[0, a]$. Hence the firm maximizes the quadratic $(a - c)q_1 - q_1^2$,from which we get the optimal production level as well as the resulting price and utility

$$q_M = \tfrac{1}{2}(a - c); \quad p_M = \tfrac{1}{2}(a + c); \quad u_M(q_M) = \tfrac{1}{4}(a - c)^2$$

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Note that if we had $a < c$ it is optimal to produce $q_M = 0$, so in general we have

$$q_M = \tfrac{1}{2}[a - c]_+.$$

The duopoly

Suppose now that Firm 2 is back and produces q_2. Then Firm 1 solves

$$\max_{q_1 \geq 0} q_1[(a - q_2) - q_1]_+ - cq_1$$

as in the monopolist case but with a replaced by $a - q_2$. A symmetric problem is solved by Firm 2. Thus, equilibrium is characterized by the equations

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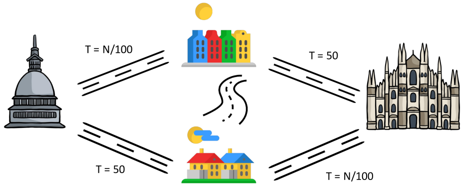
Assuming that both firms produce a positive quantity, we get the unique solution $q_1 = q_2 = \frac{1}{3}(a-c)$. The total production and the resulting price are

$$q_D = \tfrac{2}{3}(a-c) > q_M \quad ; \quad p_D = \tfrac{1}{3}(a+2c) < p_M$$

and each firm makes an utility

$$u_D = \tfrac{1}{9}(a-c)^2 < u_M.$$

Braess Paradox



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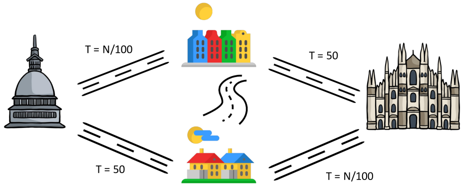

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- What are the Nash equilibria if the Up-Down street between the two small cities is closed for maintenance work?
- What if the Up-Down street is available with 5 minutes travel time?
- What if the Up-Down street can also be used Down-Up?

El Farol bar

In Santa Fe there are 500 young people, happy to go to the El Farol bar. More people in the bar, happier they are, till they reach 300 people. They can also choose to stay at home. So utility function can be assumed to be 0 if they stay at home, $u(x) = x$ if $x \leq 300$, $u(x) = 300 - x$ if $x > 300$.

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A mixed symmetric equilibrium is also present in this case.

Auctions

Several types of auctions since ancient times: sequential offers, sealed offers, first price, second price,...

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We consider only auctions where the winner is the highest bidder. In case of tie in the highest bid the winner is the one who values more the object.

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4. The two highest bids are the same and one is made by player 1. The highest bid b_1 satisfies $v_2 \leq b_1 \leq v_1$. All such bid profiles are Nash equilibria.

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3. A player's bid equalizing her evaluation is a weakly dominant strategy