Existence of Equilibria for Strategic Games

Roberto Cominetti

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Topics

- Mixed extension of finite games
- Best response maps
- Nash existence theorem
- Examples of $n$-person games
  - Hotelling game
  - Cournot competition
  - Braess paradox
  - El Farol bar
  - Auctions
Nash equilibrium in pure strategies might not exist

Consider the game

\[
\begin{pmatrix}
(4,0) & (3,1) \\
(3,5) & (5,0)
\end{pmatrix}
\]

There is no equilibrium... in pure strategies.
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There is no equilibrium... in pure strategies.

A player cannot use the same strategy all the time; this would be observed and the other player could take advantage from this.

- It makes sense to play strategies according to some probability scheme.
- But these probabilities must be chosen strategically!
Simplexes and mixed strategies

Definition

Let $A$ be a finite strategy set with $d$ elements (also called \textit{pure strategies}). The set of \textit{mixed strategies} over the set $A$ is the $d$-dimensional simplex

$$
\Sigma_A = \{ (x_a)_{a \in A} \in \mathbb{R}^A : x_a \geq 0, \sum_{a \in A} x_a = 1 \}.
$$
Simplexes and mixed strategies

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$$\Sigma_A = \{(x_a)_{a \in A} \in \mathbb{R}^A : x_a \geq 0, \sum_{a \in A} x_a = 1\}.$$

A vector $x = (x_a)_{a \in A} \in \Sigma_A$ defines a probability distribution on the set $A$ with

$$x_a = \mathbb{P}($$ playing the pure strategy $a$$)$$
Mixed extension of 2-player games

Consider a 2-person game with strategy sets $I = \{1, \ldots, n\}$ and $J = \{1, \ldots, m\}$, and payoff matrices $(A, B)$. In the mixed extension of the game player 1 choses a probability distribution $x \in \Sigma_I$ and player 2 a probability distribution $y \in \Sigma_J$. The probability of observing the outcome $ij$ is the product $x_i y_j$ and then the expected payoffs for both players are respectively:

Player 1: $f(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ij} x_i y_j = x' Ay$

Player 2: $g(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} B_{ij} x_i y_j = x' By$
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Consider a 2-person game with strategy sets $I = \{1, \ldots, n\}$ and $J = \{1, \ldots, m\}$, and payoff matrices $(A, B)$. In the mixed extension of the game player 1 chooses a probability distribution $x \in \Sigma_I$ and player 2 a probability distribution $y \in \Sigma_J$.

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Mixed extension of 2-player games

Alternatively:

- The expected payoff for player 1 when playing the pure strategy $i \in I$ against the mixed strategy $y \in \Sigma_J$ of player 2 is

$$u_i(y) = \sum_{j=1}^{m} A_{ij}y_j$$

and then

$$f(x, y) = \sum_{i=1}^{n} x_i u_i(y)$$

- The expected payoff for player 2 when playing the pure strategy $j \in J$ against the mixed strategy $x \in \Sigma_I$ of player 1 is

$$v_j(x) = \sum_{i=1}^{n} B_{ij}x_i$$

and then

$$g(x, y) = \sum_{j=1}^{m} y_j v_j(x)$$
Best responses

Nash equilibrium relies on the assumption that players maximize their payoff with respect to their own variable, taking for granted the choice of the other player.
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Player 1: $\max_{x \in \Sigma_I} \sum_{i=1}^{n} x_i u_i(y) \Rightarrow BR_1(y) = \text{Argmax}_{x \in \Sigma_I} \sum_{i=1}^{n} x_i u_i(y)$

Player 2: $\max_{y \in \Sigma_J} \sum_{j=1}^{m} y_j v_j(x) \Rightarrow BR_2(x) = \text{Argmax}_{y \in \Sigma_J} \sum_{j=1}^{m} y_j v_j(x)$
Best responses

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\text{Player 1: } & \quad \max_{x \in \Sigma_I} \sum_{i=1}^{n} x_i u_i(y) \quad \Rightarrow \quad BR_1(y) = \operatorname{Argmax}_{x \in \Sigma_I} \sum_{i=1}^{n} x_i u_i(y) \\
\text{Player 2: } & \quad \max_{y \in \Sigma_J} \sum_{j=1}^{m} y_j v_j(x) \quad \Rightarrow \quad BR_2(x) = \operatorname{Argmax}_{y \in \Sigma_J} \sum_{j=1}^{m} y_j v_j(x)
\end{align*}
\]

\(BR_1(y)\) is the set of all \(x\)’s that maximize \(f(\cdot, y)\) for a fixed \(y \in \Sigma_J\).

\(BR_2(x)\) is the set of all \(y\)’s that maximize \(g(x, \cdot)\) for a fixed \(x \in \Sigma_I\).
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Player 1: \[
\max_{x \in \Sigma_I} \sum_{i=1}^{n} x_i u_i(y) \quad \Rightarrow \quad BR_1(y) = \text{Argmax}_{x \in \Sigma_I} \sum_{i=1}^{n} x_i u_i(y)
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Player 2: \[
\max_{y \in \Sigma_J} \sum_{j=1}^{m} y_j v_j(x) \quad \Rightarrow \quad BR_2(x) = \text{Argmax}_{y \in \Sigma_J} \sum_{j=1}^{m} y_j v_j(x)
\]

$BR_1(y)$ is the set of all $x$’s that maximize $f(\cdot, y)$ for a fixed $y \in \Sigma_J$.

$BR_2(x)$ is the set of all $y$’s that maximize $g(x, \cdot)$ for a fixed $x \in \Sigma_I$.

Hence, a Nash equilibrium in mixed strategies is a pair $(\bar{x}, \bar{y}) \in \Sigma_I \times \Sigma_J$ such that

\[
\begin{cases} 
\bar{x} \in BR_1(\bar{y}) \\
\bar{y} \in BR_2(\bar{x})
\end{cases}
\]
Example: finding Nash equilibria

Consider the game

\[
\begin{pmatrix}
(4, 0) & (3, 1) \\
(3, 5) & (5, 0)
\end{pmatrix}
\]

Player 1 selects *Top* or *Bottom* with probabilities \((p, 1 - p)\).
Player 2 chooses *Left* or *Right* with probabilities \((q, 1 - q)\).
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Player 1 selects *Top* or *Bottom* with probabilities \((p, 1 - p)\).
Player 2 chooses *Left* or *Right* with probabilities \((q, 1 - q)\).

Then their expected payoffs are respectively

\[
f(p, q) = p \cdot u_T(q) + (1 - p) \cdot u_B(q)
\]

\[
= p \cdot [4q + 3(1 - q)] + (1 - p) \cdot [3q + 5(1 - q)]
\]

\[
g(p, q) = q \cdot v_L(p) + (1 - q) \cdot v_R(p)
\]

\[
= q \cdot [5(1 - p)] + (1 - q) \cdot [1p]
\]
Example: finding Nash equilibria

\[ BR_1(q) = \arg\max_{p \in [0,1]} p \, u_T(q) + (1-p) \, u_B(q) \]

\[ u_T(q) = 4q + 3(1-q) \quad ; \quad u_B(q) = 3q + 5(1-q) \]

- \( q > \frac{2}{3} \) \( \Rightarrow \) \( u_T(q) > u_B(q) \) \( \Rightarrow \) it is optimal for player 1 to choose \( p = 1 \).
- \( q < \frac{2}{3} \) \( \Rightarrow \) \( u_T(q) < u_B(q) \) \( \Rightarrow \) it is optimal for player 1 to choose \( p = 0 \).
- \( q = \frac{2}{3} \) \( \Rightarrow \) \( u_T(q) = u_B(q) \) \( \Rightarrow \) every \( p \in [0,1] \) is equally good for player 1.
Example: finding Nash equilibria

\[
BR_2(p) = \text{Argmax}_{q \in [0,1]} q \nu_L(p) + (1-q) \nu_R(p)
\]

\[
\nu_L(p) = 5(1-p) ; \quad \nu_R(p) = p
\]

- \( p < \frac{5}{6} \) \( \Rightarrow \) \( \nu_L(p) > \nu_R(p) \) \( \Rightarrow \) it is optimal for player 2 to choose \( q = 1 \).
- \( p > \frac{5}{6} \) \( \Rightarrow \) \( \nu_L(p) < \nu_R(p) \) \( \Rightarrow \) it is optimal for player 2 to choose \( q = 0 \).
- \( p = \frac{5}{6} \) \( \Rightarrow \) \( \nu_L(p) = \nu_R(p) \) \( \Rightarrow \) every \( q \in [0,1] \) is equally good for player 2.
Example: finding Nash equilibria

$$BR_1(q) = \begin{cases} 1 & \text{if } q > \frac{2}{3} \\ 0 & \text{if } q < \frac{2}{3} \\ [0, 1] & \text{if } q = \frac{2}{3} \end{cases}$$

$$BR_2(p) = \begin{cases} 1 & \text{if } p < \frac{5}{6} \\ 0 & \text{if } p > \frac{5}{6} \\ [0, 1] & \text{if } p = \frac{5}{6} \end{cases}$$

Unique Nash equilibrium: $\bar{p} = \frac{5}{6}$ and $\bar{q} = \frac{2}{3}$
General strategic games

Consider an $n$-player game with strategy sets $X_i$ and payoffs $f_i : X \rightarrow \mathbb{R}$ where as usual $X = \prod_{j=1}^{n} X_j$ is the set of strategy profiles.
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Consider an $n$-player game with strategy sets $X_i$ and payoffs $f_i : X \rightarrow \mathbb{R}$ where as usual $X = \prod_{j=1}^{n} X_j$ is the set of strategy profiles.

Each player $i = 1, \ldots, n$ maximizes her payoff with respect to her own variable $x_i \in X_i$ while taking for granted the choice of the other players $x_{-i} \in \prod_{j \neq i} X_j$. 

Define the best response map $BR_i : X_{-i} \rightarrow X_i$ as $BR_i(x_{-i}) = \{x_i \in X_i : f_i(x_i, x_{-i}) \geq f_i(z_i, x_{-i}) \quad \forall \ z_i \in X_i \}$. Then: (¯$x_i x_{-i}$)$_{i=1}^n$ is a Nash equilibrium if and only if for each player $i = 1, \ldots, n$ we have $\bar{x}_i \in BR_i(\bar{x}_{-i})$. 

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Define the best response map $BR_i : X_{-i} \rightarrow X_i$ as

$$BR_i(x_{-i}) = \{x_i \in X_i : f_i(x_i, x_{-i}) \geq f(z_i, x_{-i}) \forall z_i \in X_i\}$$

which associates to each possible strategies $x_{-i}$ of the other players the set of $x_i$'s that maximize the payoff $f(\cdot, x_{-i})$. 

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which associates to each possible strategies $x_{-i}$ of the other players the set of $x_i$'s that maximize the payoff $f(\cdot, x_{-i})$.

Then: $(\bar{x}_i)_{i=1}^n$ is a Nash equilibrium if and only if for each player $i = 1, \ldots, n$ we have $\bar{x}_i \in BR_i(\bar{x}_{-i})$. 
The Nash theorem

Theorem

Given a $n$-player game with strategy sets $X_i$ and payoff functions $f_i : X \to \mathbb{R}$ where $X = \prod_{i=1}^{n} X_i$. Suppose:

- each $X_i$ is a closed bounded convex subset in a finite dimensional space $\mathbb{R}^{d_i}$
- each $f_i : X \to \mathbb{R}$ is continuous
- $x_i \mapsto f_i(x_i, x_{-i})$ is a (quasi) concave function for each fixed $x_{-i} \in X_{-i}$

Then there exists at least one Nash equilibrium.
A simple 2-player game

Let $X = Y = [0, 10]$ and let the payoffs of the players be

\[
\begin{align*}
f(x, y) &= -x^2 - 2xy + 12x + 1 \\
g(x, y) &= -y^2 + 2xy + 8y + 7
\end{align*}
\]

The common strategy set is a closed interval, thus a closed convex bounded set. The functions are continuous, and concave in one variable when the other is fixed.
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We get $BR_1(y) = \{\max(0, 6 - y)\}$ and $BR_2(x) = \{\min(10, 4 + x)\}$.

To find an equilibrium we solve

$$
\begin{cases}
x = \max(0, 6 - y) \\
y = \min(10, 4 + x)
\end{cases}
$$

with unique solution $(\bar{x}, \bar{y}) = (1, 5)$. 

Mixed equilibria for 2-player finite games

Corollary

Every 2-player finite game has at least one Nash equilibrium in mixed strategies.
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Every 2-player finite game has at least one Nash equilibrium in mixed strategies.

In this case

- $\Sigma_I \subseteq \mathbb{R}^I$ and $\Sigma_J \subseteq \mathbb{R}^J$ are simplexes, hence closed bounded and convex
- $f(x, y) = x' Ay$ and $g(x, y) = x' By$ are jointly continuous w.r.t $(x, y)$
- $f(x, y) = \sum_{i=1}^{n} x_i u_i(y)$ is linear with respect to $x$ (for fixed $y$) and $g(x, y) = \sum_{j=1}^{m} y_j v_j(x)$ is linear with respect to $y$ (for fixed $x$)

and thus the assumptions of Nash theorem are fulfilled.
Mixed equilibria for $n$-player finite games

Consider an $n$-person finite game with strategy sets $A_i$ and payoffs $f_i(a_1, \ldots, a_n)$. In the mixed extension each player $i$ chooses a probability distribution $x^i \in \Sigma_{A_i}$, that is to say, $x^i_{a_i} \geq 0$ for all $a_i \in A_i$ and $\sum_{a_i \in A_i} x^i_{a_i} = 1$. 

Corollary

Every $n$-player finite game has at least one Nash equilibrium in mixed strategies.
Mixed equilibria for \( n \)-player finite games

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Denote \( A = \prod_{i=1}^{n} A_i \) the set of pure strategy profiles. The probability of observing an outcome \( (a_1, \ldots, a_n) \in A \) is the product \( \prod_{i=1}^{n} x^i_{a_i} \) and the expected payoffs are:

\[
\bar{f}_i(x^1, \ldots, x^n) = \sum_{(a_1, \ldots, a_n) \in A} f_i(a_1, \ldots, a_n) \prod_{j=1}^{n} x^j_{a_j} = \sum_{a_i \in A_i} x^i_{a_i} u_i(a_i, x^{-i})
\]

\[
u_i(a_i, x^{-i}) = \sum_{a_j \in A_j, j \neq i} f_i(a_1, \ldots, a_n) \prod_{j \neq i} x^j_{a_j}
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$$

Corollary

Every $n$-player finite game has at least one Nash equilibrium in mixed strategies.
Maximizing over a simplex

Consider the problem of maximizing a weighted average

\[
\text{maximize } 15x_1 + 18x_2 + 23x_3 + 23x_4 \\
\text{subject to:} \\
x_1 + x_2 + x_3 + x_4 = 1 \\
x_1, \ldots, x_4 \geq 0
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& \quad x_1, \ldots, x_4 \geq 0
\end{align*}
\]

Clearly the optimal value is 23 and is attained by putting all the weight on the variables \(x_3\) and \(x_4\) which have larger coefficients.

In particular \((0, 0, 1, 0)\) and \((0, 0, 0, 1)\) are optimal, as well as \((0, 0, \frac{1}{2}, \frac{1}{2})\).
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In particular \((0, 0, 1, 0)\) and \((0, 0, 0, 1)\) are optimal, as well as \((0, 0, \frac{1}{2}, \frac{1}{2})\).

In general the optimal solutions are all vectors \((x_1, x_2, x_3, x_4)\) with \(x_1 = x_2 = 0\) and \(x_3, x_4 \geq 0\) with \(x_3 + x_4 = 1\).
Maximizing over a simplex

Consider a player with a finite set $A$ of pure strategies. Let $u_a = u_a(x_{-i})$ be the expected payoff when playing action $a \in A$ given that the other player(s) use mixed strategies $x_{-i}$. Then the best response requires to solve

$$\max_{x \in \Sigma_A} \sum_{a \in A} x_a u_a$$

In order to render the weighted average $\sum_{a \in A} x_a u_a$ as large as possible one simply has to put all the weight on the variables with largest coefficients $u_a$. 

Note that there is always a best response in pure strategies: choose any $a$ with maximal $u_a$ and set $x_a = 1$ and $x_{a'} = 0$ for $a' \neq a$. 

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Thus, setting $\nu = \max_{a \in A} u_a$ we actually have

$$\max_{x \in \Sigma_A} \sum_{a \in A} x_a u_a = \nu$$

and $x \in \Sigma_A$ is a best response (optimal solution) if and only if

$$(\forall a \in A) \ x_a > 0 \Rightarrow u_a = \nu.$$

Note that there is always a best response in pure strategies: choose any $a$ with maximal $u_a$ and set $x_a = 1$ and $x_{a'} = 0$ for $a' \neq a$. 
Maximizing over a simplex

Definition
For a non-negative vector \((x_1, \ldots, x_n) \geq 0\) we define its support as the set of indexes of its strictly positive entries, that is

\[ spt(x) = \{i : x_i > 0\} \]
Mixed equilibria in 2-player finite games

In a 2-player finite game, the pair \((\bar{x}, \bar{y}) \in \Sigma_I \times \Sigma_J\) is a Nash equilibrium in mixed strategies iff there exists \(v, w \in \mathbb{R}\) such that

\[
\begin{align*}
\quad u_i(\bar{y}) &= v & \text{for all } i \in \text{spt}(\bar{x}) \\
\quad u_i(\bar{y}) &\leq v & \text{for all } i \notin \text{spt}(\bar{x}) \\
\quad v_j(\bar{x}) &= w & \text{for all } j \in \text{spt}(\bar{y}) \\
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  v_j(\bar{x}) = w & \text{for all } j \in \text{spt}(\bar{y}) \\
  v_j(\bar{x}) \leq w & \text{for all } j \notin \text{spt}(\bar{y})
\end{cases}
\]

In words: Player 1 should play with positive probability only the rows with maximal expected payoff $u_i(\bar{y}) = v$, and symmetrically Player 2 should assign positive probability only to the columns with maximal payoff $v_j(\bar{x}) = w$.

- $v = \text{equilibrium payoff for Player 1}$
- $w = \text{equilibrium payoff for Player 2}$
Mixed equilibria in 2-player finite games

In a 2-player finite game, the pair \((\bar{x}, \bar{y}) \in \Sigma_I \times \Sigma_J\) is a Nash equilibrium in mixed strategies iff there exists \(v, w \in \mathbb{R}\) such that

\[
\begin{align*}
\sum_{j=1}^{m} A_{ij} \bar{y}_j &= v &\text{for all } i \in \text{spt}(\bar{x}) \\
\sum_{j=1}^{m} A_{ij} \bar{y}_j &\leq v &\text{for all } i \not\in \text{spt}(\bar{x})
\end{align*}
\]

\[
\begin{align*}
\sum_{i=1}^{n} B_{ij} \bar{x}_i &= w &\text{for all } j \in \text{spt}(\bar{y}) \\
\sum_{i=1}^{n} B_{ij} \bar{x}_i &\leq w &\text{for all } j \not\in \text{spt}(\bar{y})
\end{align*}
\]

“Linear” system of equations and inequalities in \(n + m + 2\) unknowns: \(\bar{x}_i, \bar{y}_j, v, w\)

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- \(v\) = equilibrium payoff for Player 1
- \(w\) = equilibrium payoff for Player 2
Brute force algorithm

1. Guess the supports of the equilibria $spt(\tilde{x})$ and $spt(\tilde{y})$
Brute force algorithm

1. Guess the supports of the equilibria \( spt(\bar{x}) \) and \( spt(\bar{y}) \)
2. Ignore the inequalities and find \( x, y, v, w \) by solving the linear system of \( n + m + 2 \) equations

\[
\begin{align*}
\sum_{i=1}^{n} x_i &= 1 \\
\sum_{j=1}^{m} A_{ij} y_j &= v \quad \text{for all } i \in spt(\bar{x}) \\
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\end{align*}
\]
Brute force algorithm

1. Guess the supports of the equilibria $spt(\bar{x})$ and $spt(\bar{y})$

2. Ignore the inequalities and find $x, y, v, w$ by solving the linear system of $n + m + 2$ equations

\[
\begin{align*}
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\sum_{j=1}^{m} A_{ij} y_j &= v & \text{for all } i \in spt(\bar{x}) \\
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\end{align*}
\]

3. Check whether the ignored inequalities are satisfied.

If $x_i \geq 0, y_j \geq 0, \sum_{j=1}^{m} A_{ij} y_j \leq v$ and $\sum_{i=1}^{n} B_{ij} x_i \leq w$ then Stop: we have found a mixed equilibrium. Otherwise, go back to step 1 and try another guess of the supports.
Exercises

1. Find all the equilibria for the Battle of the sexes

\[
\begin{pmatrix}
3, 2 & 1, 1 \\
0, 0 & 2, 3
\end{pmatrix}
\]

2. Find all the equilibria for

- Hawks & Doves
- Crossing game
- Prisoner’s Dilemma
- Tragedy of the Commons
- Rock-Scissors-Paper
Lemke-Howson Algorithm

Enumerating all the possible supports in the brute force algorithm quickly becomes computationally prohibitive: there are potentially \((2^n - 1)(2^m - 1)\) options!

For \(n \times n\) games the number of combinations grow very quickly

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Lemke-Howson proposed a more efficient algorithm... though still with exponential running time in the worst case.

**Exercise:** Search Lemke-Howson’s algorithm on the web, download it, and use it to solve the previous games.
Hotelling game

Several icecream vendors must place their cart in a beach 1 kilometer long. People are distributed uniformly on the beach and choose the closest cart to get icecream. People having several carts at the same distance are split evenly.
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4. With six or more carts there are infinitely many equilibria.
Cournot competition

Two firms choose to produce a certain good, with a unit production cost of \( c > 0 \). Firm 1 produces a quantity \( q_1 \), whereas firm 2 produces a quantity \( q_2 \).
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Therefore, the utilities of both firms are respectively

$$u_1(q_1, q_2) = q_1 p(q_1 + q_2) - c q_1$$
$$u_2(q_1, q_2) = q_2 p(q_1 + q_2) - c q_2$$
The monopolist

Suppose that Firm 2 is temporarily out of business, so that \( q_2 = 0 \). Then Firm 1 becomes a monopolist and maximizes its utility

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q_M = \frac{1}{2}(a - c); \quad p_M = \frac{1}{2}(a + c); \quad u_M(q_M) = \frac{1}{4}(a - c)^2
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Note that if we had $a < c$ it is optimal to produce $q_M = 0$, so in general we have

$$q_M = \frac{1}{2}[a - c]_+.$$
The duopoly

Suppose now that Firm 2 is back and produces \( q_2 \). Then Firm 1 solves

\[
\max_{q_1 \geq 0} q_1 [(a - q_2) - q_1]_+ - cq_1
\]

as in the monopolist case but with \( a \) replaced by \( a - q_2 \). A symmetric problem is solved by Firm 2. Thus, equilibrium is characterized by the equations

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q_1 = \frac{1}{2} [a - q_2 - c]_+ \\
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Assuming that both firms produce a positive quantity, we get the unique solution \( q_1 = q_2 = \frac{1}{3} (a - c) \). The total production and the resulting price are

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q_D = \frac{2}{3} (a - c) > q_M; \\
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Braess Paradox

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- What are the Nash equilibria if the Up-Down street between the two small cities is closed for maintenance work?
- What if the Up-Down street is available with 5 minutes travel time?
- What if the Up-Down street can also be used Down-Up?
El Farol bar

In Santa Fe there are 500 young people, happy to go to the El Farol bar. More people in the bar, happier they are, till they reach 300 people. They can also choose to stay at home. So utility function can be assumed to be 0 if they stay at home, \( u(x) = x \) if \( x \leq 300 \), \( u(x) = 300 - x \) if \( x > 300 \).
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A mixed symmetric equilibrium is also present in this case.
Auctions

Several types of auctions since ancient times: sequential offers, sealed offers, first price, second price, . . .

- There are \( n \) bidders, each one has a valuation \( v \) for the object, which is kept as private information. We assume that \( v_1 > v_2 > \cdots > v_n \).
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We consider only auctions where the winner is the highest bidder. In case of tie in the highest bid the winner is the one who values more the object.
First price auction

In a first price auction the rule is: the player $i$ offering the highest bid $b_i$ gets the object and pays exactly her bid. The other players pay nothing.
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2. One Nash equilibrium is $(v_2, v_2, v_3, \ldots, v_n)$.
3. In all equilibria the winner is player 1.
4. The two highest bids are the same and one is made by player 1. The highest bid $b_1$ satisfies $v_2 \leq b_1 \leq v_1$. All such bid profiles are Nash equilibria.
Second price auctions

In a second price auction the rule is: the player $i$ offering the highest bid $b_i$ gets the object and pays the second highest bid. The other players pay nothing,
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3. A player’s bid equalizing her evaluation is a weakly dominant strategy