Games in Strategic Form

Roberto Cominetti

LUISS

Topics

- Rationality and utility theory
- Players, strategies, payoffs
- Games in strategic form
- Nash equilibrium
- Elimination of dominated strategies
- Finite 2-person games
- Examples of 2-person games:
  - prisoner's dilemma
  - tragedy of the commons
  - battle of the sexes
  - doves and hawks...

Setting

1. Optimization: one decision maker

$$\max_{x \in X} f(x)$$

2. Games: at least two decision makers

$$\max_{x \in X} f(x,y) \qquad ; \qquad \max_{y \in Y} g(y,x)$$

Crucial difference: The best to do is easily definable with one decision maker. More difficult with many decision makers. Several variants depending on context:

- game theory: competition/cooperation
- social choice
- mechanism design

Loose description of a game

A process that can be described by:

1. A set of players... more than one
2. An initial situation
3. The way the players must act and all their available moves
4. All possible final situations
5. The preferences of all agents on the set of the final situations

Examples:

- The chess game
- Two people bargaining on how to divide a pie

Modeling a game

A game is modeled by specifying

- The set of players
- Their strategies
- Their preferences on all possible outcomes of the game

A strategy for a player: the specification of an action at any time she could be called to make a move

Assumptions of the theory

Players are

1. Selfish
2. Rational

Selfishness means that players care only about their own preferences on the outcomes of the game. This is NOT an ethical issue, but a mathematical assumption. A player's preference may depend on other player's results and include elements of envy and/or altruism.

Rationality is a much more involved issue.

Preferences

Definition

Let X be a set. A preference relation on X is a binary relation $\succeq$ fulfilling, for all $x, y, z \in X$:

1) $x \succeq x$ (*reflexivity*)
2) *either* $x \succeq y$ *or* $y \succeq x$ *or both* (*completeness*)
3) $x \succeq y$ *and* $y \succeq z$ *imply* $x \succeq z$ (*transitivity*)

First rationality assumption: Each player is able to provide a preference relation over the outcomes of the game.

Utility functions

Definition

Let $\succeq$ be a preference relation over X. A utility function representing $\succeq$ is a function $u : X \to \mathbb{R}$ such that

$$x \succeq y \iff u(x) \geq u(y).$$

Second rationality assumption: Each player is able to provide a utility function representing her preferences.

A utility function need not exist, however it exists under fairly general conditions, in particular if X is a finite set. When it exists, it is not unique.

Equivalent utilities

If $u(\cdot)$ is a utility function of a player, then $g \circ u$ is another utility function, for every strictly increasing function g.

Thus $u^3(\cdot), \exp(u(\cdot)), \arctan(u(\cdot))$ are also utility functions inducing the same preference.

Given a utility function $u(\cdot)$, a widely used transformation is an affine transformation $v(\cdot) = au(\cdot) + b$, where $a > 0$, b any real.

Why utility functions ?

Utility functions not only serve to express preferences but also to express the intensity of such preferences. They are particularly useful when some random choice is present in the game.

Example: The simplest game of chance

Will you play with me if the game is the following?

- We toss a coin. If the result is Heads you take 1 Euro from me, if the result is Tails you give me 2 Euro. Expected gain $= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-2) = -0.5$

What about the following variant?

- We toss a coin. If the result is Heads you take 100 Euro from me, otherwise I take 90 from you? Expected gain $= \frac{1}{2} \cdot 100 + \frac{1}{2} \cdot (-90) = 5$

In both games you prefer Heads to Tails, but preferences are not all that goes into these games. We need to calculate expected utilities.

Probability issues and Decision theory assumptions

Third rationality assumption: The players use consistently the probability laws, in particular they are consistent w.r.t. calculation of expected utilities, they are able to update probabilities according to Bayes rule. . .

Fourth rationality assumption: Each player is able to use decision theory, whenever it is possible. This means that the players are utility maximizers and/or cost minimizers.

Summarizing the rationality assumptions

1. The players are able to consistently rank the outcomes of the game
2. The players are able to provide a utility function for their ranking
3. The players apply the expected value principle to build their utility functions in presence of random events
4. The players use the apparatus of decision theory anytime it is possible

Fifth rationality assumption: The basic data of the game are common knowledge and each player knows that all players are rational.

- Every player knows the strategies and utility functions of the other players.
- Not only a player knows that the other players are rational, but also that they know that he knows this, and they know that he knows that they know, etc...

Elimination of strictly dominated strategies

A basic consequence of the decision theory assumption is:

A player will not choose a strategy *a* if she has available a strategy *b* providing her a strictly higher utility, *no matter what the other players do.*

After an exam, player one strategy set is $\{18, \ldots, 30\}$, player two strategy set is $\{\text{accept}, \text{refuse}\}$. If player two preference is passing the exam with any grade, rather than repeating it, the action *refuse* is strictly dominated.

Observe, both players know this. Thus asking for one or two extra points is useless (and would change the game rules. . .)

Games in strategic form

Definition

A 2-player non-cooperative game in strategic form is given by

- *Strategy sets:* X *(player 1) and* Y *(player 2)*
- *Payoff functions:* $f : X \times Y \to \mathbb{R}$ *(player 1) and* $g : X \times Y \to \mathbb{R}$ *(player 2)*

Natural extension to n-players:

- Set of players $i \in \{1, \ldots, n\}$
- Strategy set X_i for each player $i = 1, \ldots, n$
- Strategy profile $(x_1, \ldots, x_n)$ with $x_i \in X_i$ for each $i = 1, \ldots, n$
- Set of strategy profiles $X = \prod_{i=1}^{n} X_i$
- Payoff functions $f_i : X \to \mathbb{R}$ for each player $i = 1, \ldots, n$

Nash equilibrium

A Nash equilibrium for the 2-player strategic game is a pair $(\bar{x}, \bar{y}) \in X \times Y$ with:

- $f(\bar{x}, \bar{y}) \geq f(x, \bar{y})$ for all $x \in X$
- $g(\bar{x}, \bar{y}) \geq g(\bar{x}, y)$ for all $y \in Y$

Extension to n-player games: A strategy profile $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ such that, for each player $i = 1, \ldots, n$ we have

$$f_i(\bar{x}) \geq f(x_i, \bar{x}_{-i}) \text{ for all } x_i \in X_i$$

where $(x_i, \bar{x}_{-i})$ is the strategy profile $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_n)$.

In words: no player i has incentive to deviate by choosing a different $x_i \neq \bar{x}_i$.

Nash equilibrium and dominated strategies

Suppose $\bar{x} \in X$ is a (weakly) dominant strategy for P1, that is to say, for each $x \in X$ we have:

$$f(\bar{x}, y) \geq f(x, y) \text{ for all } y \in Y.$$

Then, if $\bar{y}$ maximizes the function $g(\bar{x}, \cdot)$, then $(\bar{x}, \bar{y})$ is a Nash equilibrium.

Finite 2-player games

A two player game, where one player, called row player, has n strategies and the second, called column player, has m strategies, can be represented by a pair of $n \times m$ matrices, denoted (A, B), often called bimatrix game:

$$\begin{pmatrix} (a_{11}, b_{11}) & \dots & (a_{1j}, b_{1j}) & \dots & (a_{1m}, b_{1m}) \\ \dots & \dots & \dots & \dots & \dots \\ (a_{i1}, b_{i1}) & \dots & (a_{ij}, b_{ij}) & \dots & (a_{im}, b_{im}) \\ \dots & \dots & \dots & \dots & \dots \\ (a_{n1}, b_{n1}) & \dots & (a_{nj}, b_{nj}) & \dots & (a_{nm}, b_{nm}) \end{pmatrix}.$$

- The strategy spaces of the players are $X = \{1, \dots, n\}$ and $Y = \{1, \dots, m\}$
- The choices of $i \in X$ and $j \in Y$ yield the outcome ij
- The utilities of the players on the outcome ij are a_{ij} and b_{ij} respectively.

An example

The two players have two strategies each:

$$\begin{pmatrix} (8,8) & (2,7) \\ (7,2) & (0,4) \end{pmatrix}$$

Utilities of player 1:

$$\begin{pmatrix} 8 & 2 \\ 7 & 0 \end{pmatrix}$$

The second row is strictly dominated by the first, thus player 1 must play top.

What should then be the choice of the second player? Left or right?

First examples

$$\begin{pmatrix} (3,2) & (1,1) \\ (1,0) & (2,1) \end{pmatrix}$$

Nash equilibria: (first row, first column) payoffs $(3,2)$ and (second row, second column) payoffs $(2,1)$. It is likely that the players will agree on the first.

$$\begin{pmatrix} (3,2) & (0,0) \\ (0,0) & (2,3) \end{pmatrix}$$

Nash equilibria: (first row, first column) payoffs $(3,2)$ and (second row, second column) payoffs $(2,3)$. Note that the players have opposite preferences on the two outcomes

$$\begin{pmatrix} (0,0) & (1,1) \\ (1,1) & (0,0) \end{pmatrix}$$

Nash equilibria: (first row, second column) payoffs $(1,1)$ and (second row, second column) payoffs $(1,1)$. The players are indifferent on the two outcomes, but need coordination to fall in one of them.

Another example

$$\begin{pmatrix} (8,8) & (2,7) & (4,10) \\ (7,2) & (0,4) & (3,0) \\ (5,3) & (3,9) & (10,4) \end{pmatrix}$$

- The first player can eliminate ...
- Knowing this the second can eliminate ...
- Knowing this the first can eliminate ...
- The outcome is

Prisoner's Dilemma I

The most famous example. Two prisoners are facing a trial.

- If they keep silent they are both charged for a minor offense and get a sentence of 1 year in prison each (because of lack of evidence).
- If one decides to confess he is released and the other gets 6 years in jail.
- If both confess each one gets 5 years in jail.

$$\begin{pmatrix} (-1,-1) & (-6,0) \\ (0,-6) & (-5,-5) \end{pmatrix}$$

The unique rational outcome is not only Nash equilibrium, but also obtained with elimination of strictly dominated strategies!

Prisoner's Dilemma II

A father tells to his two children: Do you want me to give 1 Euro to you, or that I give 10 Euros to your brother?

$$\begin{pmatrix} (10,10) & (0,11) \\ (11,0) & (1,1) \end{pmatrix}$$

This game has exactly the same structure as the prisoner's dilemma.

Tragedy of the commons

This game is the two player version of a game known under the name of tragedy of commons

$$\begin{pmatrix} (a,a) & (b,c) \\ (c,b) & (d,d) \end{pmatrix}$$

with $c > a > d > b$.

Standard situation when two people exploit common resources: it is a strictly dominating strategy to try to use them as much as possible, but since they are finite this makes the situation bad for all.

Battle of the sexes

A couple plans to spend Sunday together, either by going to watch the Roma vs Lazio match, or to the movies to watch a recently released film. She loves movies while he prefers soccer, but both prefer being together rather than by their own.

$$\begin{pmatrix} (3,2) & (1,1) \\ (0,0) & (2,3) \end{pmatrix}$$

Which are the equilibria?

Higher payoffs need not be better...

Compare the following games

The first game:

$$\begin{pmatrix} (10,10) & (3,15) \\ (15,3) & (5,5) \end{pmatrix}$$

The second one:

$$\begin{pmatrix} (8,8) & (2,7) \\ (7,2) & (0,4) \end{pmatrix}$$

Observe: In any outcome the players are better off in the first game rather than in the second. Yet, it is more convenient for them to play the second !

More strategies might also hurt...

The first game:

$$\begin{pmatrix} (10,10) & (3,5) \\ (5,3) & (1,1) \end{pmatrix}$$

The second game, containing all possible outcomes the first, and some further outcomes:

$$\begin{pmatrix} (1,1) & (11,0) & (4,0) \\ (0,11) & (10,10) & (3,5) \\ (0,4) & (5,3) & (1,1) \end{pmatrix}$$

Having less available actions can make the players better off!

Hawks and Doves

Two fighting birds can behave either aggressive as a Hawk or gentle as a Dove.

- If both are Hawks they get badly hurt and each one looses -100 utils.
- If both are Doves they get 1 util each.
- If a Hawk meets a Dove, the Hawk gets 10 and the Dove nothing.

$$\begin{pmatrix} (-100, -100) & (10, 0) \\ (0, 10) & (1, 1) \end{pmatrix}$$

There are two (equally likely) Nash equilibria in pure strategies.

Crossing game

Two drivers face a non-signalled intersection. If they both cross the crash is unavoidable. If one crosses and the other lends the first gets a slightly higher utility. If they both lend the passage they can wait forever...

$$\begin{pmatrix} (-100,-100) & (2,1) \\ (1,2) & (0,0) \end{pmatrix}$$

Again there are two (equally likely) Nash equilibria in pure strategies.

Rock-Scissors-Paper

A 2-person game is called *zero-sum* if $a_{ij} + b_{ij} = 0$ for all i, j.

Clearly in such a case it suffices to display the payoff of player 1. In the popular game Rock-Scissors-Paper the payoff matrix is

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

In this case there is no equilibrium... at least in pure strategies...