Linear Programming Duality

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Example: the dual of the production planning problem

A small manufacturer produces tools of types A and B, that require different amounts of labor, wood and metal, and yield different profits

Tool type	Labor [hr/unit]	Wood [kg/unit]	Metal [kg/unit]	Profit [€/unit]
A	1.0	1.0	2.0	50
B	2.0	1.0	1.0	40

Available are 120 hours of labor, 70 units of wood, and 100 units of metal.
The optimal production mix $x_A^* = 30$ / $x_B^* = 40$ was found by solving

$$\left\{ \begin{array}{lll} \max & 50\, x_A + 40\, x_B & \\ \text{s.t.} & x_A + 2\, x_B & \leq 120 \\ & x_A + x_B & \leq 70 \\ & 2\, x_A + x_B & \leq 100 \\ & x_A \geq 0, x_B \geq 0 & \end{array} \right.$$

Example: the dual of the production planning problem

Imagine that we want to make a takeover operation to buy all the resources held by the manufacturer: 120[h] of labor, 70[kg] of wood, 100[kg] of metal.

What prices should we offer so that our proposal is acceptable?

Are 5 [€/hr] for labor, 5 [€/kg] for wood, 10 [€/kg] for metal, fair prices?

Note that each unit of Tool A yields €50 of revenue to the manufacturer. If instead she sells the resources needed to produce that unit, she would only get

1 unit of Tool A		
1[h] of labor	=	€5
1[kg] of wood	=	€5
2[kg] of metal	=	€20
Selling revenue	=	€30

Not a good deal... she will certainly reject our offer !
She is better off by keeping the resources and use them to produce tools.

Example: the dual of the production planning problem

What if we increase the price we offer for metal to 20 [€/kg]?

In this case selling the resources required for one unit of Tool A would give

1 UNIT OF TOOL A		
1[h] of labor	=	€5
1[kg] of wood	=	€5
2[kg] of metal	=	€40
Selling revenue	=	€50

The manufaturer will now be indifferent between keeping the resources and use them to produce a Tool A, or selling the materials. If we increase any of the prices, even by a tiny amount, the manufacturer will be strictly better off by selling.

In general, if we set prices y_l, y_w, y_m our offer will be acceptable as long as

$$1\, y_l + 1\, y_w + 2\, y_m \geq 50$$

Is this enough? What about Tool B?

Example: the dual of the production planning problem

What about Tool B? At 5 [€/hr] for labor, 5 [€/kg] for wood, 20 [€/kg] for metal, the selling revenue per unit of Tool B would be

1 unit of Tool B		
2[h] of labor	=	€10
1[kg] of wood	=	€5
1[kg] of metal	=	€20
Selling revenue	=	€35

Not a good deal... our offer will still be rejected!

Our proposed prices y_l, y_w, y_m should also be competitive in terms of the revenue provided by each unit of Tool B, which translates into

$$2\,y_l + 1\,y_w + 1\,y_m \geq 40$$

Example: the dual of the production planning problem

In summary, our proposed prices must be non-negative and satisfy both

$$\begin{aligned} y_l + y_w + 2\,y_m &\geq 50 \\ 2\,y_l + y_w + y_m &\geq 40 \end{aligned}$$

Since there are many set of prices that meet these conditions, we might ask which ones are the most convenient from our perspective as buyers?

Since we want to buy all the resources: 120[h] of labor, 70[kg] of wood, 100[kg] of metal, our final bill would be

$$120y_l + 70y_w + 100y_m$$

which leads us naturally to consider the following associated dual linear program

$$\left\{ \begin{array}{ll} \min & 120y_l + 70y_w + 100y_m \\ \text{s.t.} & y_l + y_w + 2\,y_m \quad \geq 50 \\ & 2\,y_l + y_w + y_m \quad \geq 40 \\ & y_l \geq 0, y_w \geq 0, y_m \geq 0 \end{array} \right.$$

Example: the dual of the production planning problem

Primal
$$\begin{cases} \max & 50\,x_A + 40\,x_B \\ \text{s.t.} & x_A + 2\,x_B \leq 120 \quad \leftarrow y_l \text{ (labor)} \\ & x_A + x_B \leq 70 \quad \leftarrow y_w \text{ (wood)} \\ & 2\,x_A + x_B \leq 100 \quad \leftarrow y_m \text{ (metal)} \\ & x_A \geq 0, x_B \geq 0 \end{cases}$$

Dual
$$\begin{cases} \min & 120 y_l + 70 y_w + 100 y_m \\ \text{s.t.} & y_l + y_w + 2\,y_m \geq 50 \\ & 2\,y_l + y_w + y_m \geq 40 \\ & y_l \geq 0, y_w \geq 0, y_m \geq 0 \end{cases}$$

- Each resource constraint in the primal gets associated a dual variable (price).
- The right hand side coefficients in the primal become the coefficients of the dual objective function, and vice-versa.
- The constraint matrix in the dual is just the transpose of the primal matrix.
- The constraint inequalities are reversed.

Example: The dual of the production planning problem

Economic interpretation: the dual problem computes prices which reflect the *value* that each resource contributes as part of the production process, and which is derived implicitly from the prices of the final products. Only scarce resources get positive prices, whereas over-abundant resources will have their prices set to 0.

Alternatively, dual prices measure the impact in total revenue when an additional unit of the resource becomes available. If market prices are below the dual prices, then one could make a net profit by buying more resources at those market prices and putting them to work in the production process.

In other words, in a perfect market the prices that one should observe should match those predicted by the dual problem, and reflect exactly the value of each resource as an input for all the production processes for which it is relevant.

Example: the dual of the production planning problem

Algebraic interpretation: another way to interpret the dual comes from the observation that every dual feasible solution provides an upper bound for the optimal revenue v_p that the manufaturer can achieve.

Indeed, let $x \geq 0$ and $y \geq 0$ be primal and dual feasible solutions respectively. Multiplying each resource inequality by the corresponding price and summing

$$(x_A + 2x_B)y_l + (x_A + x_B)y_w + (2x_A + x_B)y_m \leq 120y_l + 70y_w + 100y_m$$

Rearraging the right hand side we have

$$x_A(y_l + y_w + 2y_m) + x_B(2y_l + y_w + y_m) \leq 120y_l + 70y_w + 100y_m$$

and then dual feasibility yields

$$50x_A + 40x_B \leq 120y_l + 70y_w + 100y_m$$

Hence $120y_l + 70y_w + 100y_m$ is an upper bound for the optimal revenue v_p.

The dual problem is therefore computing the smallest of such upper bounds.

Dual linear programs: Canonical Form

$$(P_C)\quad \begin{cases} \max c^t x \\ Ax \leq b \\ x \geq 0 \end{cases} \qquad\qquad (D_C)\quad \begin{cases} \min b^t y \\ A^t y \geq c \\ y \geq 0 \end{cases}$$

Exercise (The dual of the dual is the primal)

Rewrite (D_C) *as a maximization problem and show that its dual is* (P_C)*.*

Dual linear programs: Standard Form

$$(P_S) \quad \begin{cases} \max c^t x \\ Ax = b \\ x \geq 0 \end{cases} \qquad\qquad (D_S) \quad \begin{cases} \min b^t y \\ A^t y \geq c \end{cases}$$

Exercise (The duals in canonical and standard forms are equivalent)

1. *Transform (P_S) into canonical form; formulate the dual of the latter; and then show that this is equivalent to (D_S).*
2. *Starting from the canonical form (P_C); transform it into standard form and check that the corresponding dual is equivalent to (D_C).*

Feasibility of dual programs

Simple examples show that, given two problems in duality, the following three situations may occur:

- Both problems have feasible solutions
- Exactly one of them is feasible
- Both of them are infeasible

Example 1

Consider the primal problem

$$(P)\begin{cases} \max x_1 + 3x_2 \\ x_1 + 2x_2 \leq 7 \\ x \geq 0 \end{cases}$$

and its dual

$$(D)\begin{cases} \min 7y \\ y \geq 1 \\ 2y \geq 3 \\ y \geq 0 \end{cases}$$

Since $x = (0,0)$ fulfills the constraints of the primal problem, and $y = 2$ satisfies the dual constraints, both problems are feasible.

Example 2

Consider the primal program

$$(P)\begin{cases} \max x_1 - x_2 \\ x_1 + x_2 \leq 1 \\ -x_1 - x_2 \leq -2 \\ x \geq 0 \end{cases}$$

and its dual

$$(D)\begin{cases} \min y_1 - 2y_2 \\ y_1 - y_2 \geq 1 \\ y_1 - y_2 \geq -1 \\ y \geq 0 \end{cases}$$

The primal is infeasible (can you see why?) while $y = (1, 0)$ is feasible for the dual.

Exercise

Find an example where the primal is feasible but not the dual, and another example where they are both infeasible.

Weak duality theorem

Theorem

Let v_p be the value of the primal maximization problem and v_d the value of the dual minimization problem. Then

$$v_p \leq v_d$$

Proof

Canonical form: for each $x \geq 0$ primal-feasible we have $Ax \leq b$ so that for $y \geq 0$ dual-feasible we get the inequalities

$$b^t y \geq (Ax)^t y = x^t A^t y \geq x^t c = c^t x$$

Since this is true for all admissible x and y the result follows.

Standard form: for all $x \geq 0$ primal-feasible and y dual-feasible we have

$$b^t y = (Ax)^t y = x^t A^t y \geq x^t c = c^t x.$$

∎

Strong duality theorem

Theorem

- *If the primal and dual problems are feasible, then both problems have optimal solutions* $\bar{x}, \bar{y}$ *and the optimal values coincide*

$$v_p = c^t \bar{x} = b^t \bar{y} = v_d.$$

  In this case we say that there is no duality gap.
- *If the primal is feasible and the dual is infeasible, then* $v_p = v_d = \infty$
- *If the primal is infeasible and the dual is feasible, then* $v_p = v_d = -\infty$
- *If both the primal and the dual are infeasible, then* $v_p = -\infty < v_d = \infty$

Corollary

If the primal problem is feasible and has an optimal solution, then also the dual problem is feasible and has solutions. Moreover there is no duality gap.

Complementarity conditions: Canonical Form

$$(P)\quad \begin{cases} \max c^t x \\ Ax \leq b, x \geq 0 \end{cases} \qquad ; \qquad (D)\quad \begin{cases} \min b^t y \\ A^t y \geq c, y \geq 0 \end{cases}$$

An inequality constraint is called *tight* (*binding*, *active*) if it holds as an equality.

Theorem

Let $\bar{x}, \bar{y}$ be feasible solutions for the primal and dual respectively. Then $\bar{x}, \bar{y}$ are simultaneously optimal solutions if and only if

a) *If a primal variable is positive $\bar{x}_i > 0$, then the i-th dual constraint is tight.*

b) *If a dual variable is positive $\bar{y}_j > 0$, then the j-th primal constraint is tight.*

Proof: Since $c^t x \leq y^t Ax \leq b^t y$ it follows that $\bar{x}, \bar{y}$ are optimal iff

$$c^t \bar{x} = \bar{y}^t A\bar{x} = b^t \bar{y}$$

This equality can be written both as $\bar{x}^t(A^t\bar{y} - c) = 0$ or in the form $\bar{y}^t(A\bar{x} - b) = 0$. Since $\bar{x}, \bar{y} \geq 0$ and $A\bar{x} \leq b, A^t\bar{y} \geq c$ the latter are equivalent to (CC). ∎

An example

Consider the pair of dual linear programs

$$(P)\begin{cases} \max 2x_1 - 2x_2 \\ 2x_1 - x_2 \leq 1 \\ x_1 - 2x_2 \leq 1 \\ x \geq 0 \end{cases}$$

$$(D)\begin{cases} \min y_1 + y_2 \\ 2y_1 + y_2 \geq 2 \\ y_1 + 2y_2 \leq 2 \\ y \geq 0 \end{cases}$$

We have $v_p = v_d = 1$ with optimal solutions $\bar{x} = (\frac{1}{2}, 0)$ and $\bar{y} = (1, 0)$.

Check of the complementarity conditions:

$$\bar{x}_1 = \tfrac{1}{2} > 0 \quad \Rightarrow \quad 2\bar{y}_1 + \bar{y}_2 = 2$$
$$\bar{y}_1 = 1 > 0 \quad \Rightarrow \quad 2x_1 - x_2 = 1$$

Another example

$$(P)\begin{cases} \max x_1 - 3x_2 - 4x_3 \\ x_1 - x_3 \leq 1 \\ -x_2 - x_3 \leq 4 \\ x \geq 0 \end{cases}$$

$$(D)\begin{cases} \min \;\; y_1 + 4y_2 \\ y_1 \geq 1 \\ y_2 \leq 3 \\ y_1 + y_2 \leq 4 \\ y \geq 0 \end{cases}$$

The common optimal value of the problems is $v_p = v_d = 1$.
Optimal solutions: for the primal $x = (1, 0, 0)$ and for the dual $y = (1, 0)$.

Exercise

Check the complementarity conditions.