Linear Programming

Roberto Cominetti

LUISS

Contents of the week

- Some prototypical examples of linear programs
- Solving LP's with Julia/JuMP
- LPs in standard and canonical forms
- Polyhedra and polytopes
- Extreme points and basic feasible solutions
- Optimal solutions
- Informal description of the Simplex method

Linear programming problems

Definition

A linear program consists in maximizing or minimizing a linear function under a set of linear equality and inequality constraints

Example: Production planning

A small manufacturer produces tools of types A and B, which require different amounts of labor, wood and metal, and yield different profits as shown below

Tool type	Labor [hr/unit]	Wood [kg/unit]	Metal [kg/unit]	Profit [€/unit]
A	1.0	1.0	2.0	50
B	2.0	1.0	1.0	40

Available are 120 hours of labor, 70 units of wood, and 100 units of metal. What is the production mix that maximizes profit ?

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B	2.0	1.0	1.0	40

Available are 120 hours of labor, 70 units of wood, and 100 units of metal. What is the production mix that maximizes profit ?

$$\left\{\begin{array}{lll} \max & 50\,x_A + 40\,x_B & \\ \text{s.t.} & x_A + 2\,x_B & \leq 120 \\ & x_A + x_B & \leq 70 \\ & 2\,x_A + x_B & \leq 100 \\ & x_A \geq 0, x_B \geq 0 & \end{array}\right.$$

Example: Production planning

$$
\begin{cases}
\max & 50\,x_A + 40\,x_B \\
\text{s.t.} & x_A + 2\,x_B \leq 120 \\
 & x_A + x_B \leq 70 \\
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 & x_A \geq 0 \\
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\end{cases}
$$

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\end{cases}
$$

Optimal solution: $x_A^* = 30 \;/\; x_B^* = 40$

- Maximum profit is €3100
- Total labor used is 110 [hr]
- Total wood used is 70 [kg]
- Total metal used is 100 [kg]

Example: Diet problem

Given a set of foods with their nutrient information and cost per serving, determine the number of meal servings that satisfy a set of nutritional requirements at minimum cost.

Food	Calories [cal]	Vitamin A [mcg]	Cost [€]
Carrots	40	450	0.12
Milk	63	150	0.15
Chicken	220	5	2.00

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Suppose that the maximum number of servings is 10 for each food, and there are restrictions on the daily calories (between 1600 and 2500 [cal]) and the amount of Vitamin A (between 800 and 1100 [mcg]).

Example: Diet problem

$$\begin{cases} \min & 0.12\,x_1 + 0.15\,x_2 + 2.0\,x_3 \\ \text{s.t.} & 1600 \leq 40\,x_1 + 63\,x_2 + 220\,x_3 \leq 2500 \\ & 800 \leq 450\,x_1 + 150\,x_2 + 5\,x_3 \leq 1100 \\ & 0 \leq x_1, x_2, x_3 \leq 10 \end{cases}$$

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 & 0 \leq x_1, x_2, x_3 \leq 10
\end{cases}
$$

Optimal solution: $x_1^* = 0.0$ (carrot) / $x_2^* = 7.16$ (milk) / $x_3^* = 5.22$ (chicken)

- Minimum cost is €11.519
- Total number of Calories is 1600
- Total amount of Vitamin A is 1100

Example: Optimal transport

A gadget manufacturer has two factories F_1 and F_2 with production capacities 6 and 9, from which it serves three retail centers C_1, C_2, C_3 with demands 8, 5, 2. The transportation costs are shown below

	C_1	C_2	C_3
F_1	5	5	3
F_2	6	4	1

Which demand should be served from each factory?

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	C_1	C_2	C_3
F_1	5	5	3
F_2	6	4	1

Which demand should be served from each factory?
Amount shipped from factory F_i to retail center $C_j = x_{i,j}$

$$\begin{cases} \min & 5\,x_{1,1} + 5\,x_{1,2} + 3\,x_{1,3} + 6\,x_{2,1} + 4\,x_{2,2} + x_{2,3} \\ \text{s.t.} & x_{1,1} + x_{1,2} + x_{1,3} = 6 \\ & x_{2,1} + x_{2,2} + x_{2,3} = 9 \\ & x_{1,1} + x_{2,1} = 8 \\ & x_{1,2} + x_{2,2} = 5 \\ & x_{1,3} + x_{2,3} = 2 \\ & x_{i,j} \geq 0 \quad (\forall i = 1,2)(\forall j = 1,2,3) \end{cases}$$

Matrix notation for linear programs

A linear program with only inequality constraints

$$\begin{cases} \max \sum_{i=1}^{n} c_i x_i & \\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \leq b_1 \\ \quad\vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \leq b_m \end{cases}$$

can be written in compact form using matrix notation

$$\begin{cases} \max c^t x \\ Ax \leq b \end{cases}$$

where A is an $m \times n$ matrix, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

Matrix notation for linear programs

Similarly, a linear program with only equality plus sign constraints

$$\begin{cases} \min \sum_{i=1}^{n} c_i x_i \\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \quad = b_1 \\ \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \quad = b_m \\ x_1 \geq 0, \ldots, x_n \geq 0 \end{cases}$$

can be expressed in compact form as

$$\begin{cases} \min c^t x \\ Ax = b \\ x \geq 0 \end{cases}$$

where A is an $m \times n$ matrix, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

Transformations of linear programs

There are several forms of linear programs, each of which can be transformed into another equivalent form by using the following simple tricks:

- A minimization problem is converted to maximization (and vice-versa) by changing the sing of objective function.
- An inequality $\geq$ is converted to $\leq$ by multiplying by -1.
- A sign constraint $x_i \geq 0$ can be included as an additional row in $Ax \leq b$.
- An unconstrained variable $x_i \in \mathbb{R}$ can be replaced by $x_i = x_i^+ - x_i^-$ with $x_i^+ \geq 0$ and $x_i^- \geq 0$.
- An inequality $a^t x \leq b$ can be converted to an equality $a^t x + s = b$ plus an additional sign constraint $s \geq 0$. Here s is called a *slack variable*.
- An equality $a^t x = b$ is equivalent to two inequalities $a^t x \leq b$ and $a^t x \geq b$. Alternatively, an equality can be used to eliminate one variable from the problem.

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Transformations of linear programs

With these simple transformations every linear program can be stated in any of the following forms, for appropriately defined A, b, and c.

$$\text{Compact Form} \quad \begin{cases} \max c^t x \\ Ax \leq b \end{cases}$$

$$\text{Canonical Form} \quad \begin{cases} \max c^t x \\ Ax \leq b \\ x \geq 0 \end{cases}$$

$$\text{Standard Form} \quad \begin{cases} \max c^t x \\ Ax = b \\ x \geq 0 \end{cases}$$

Feasibility, polyhedra and polytopes

The set of vectors satisfying the equalities and inequalities of a linear program is called its *feasible set*.

These are a special class of convex sets called *polyhedra*. When they are bounded they are called *polytopes*.

Example:

$$2x_1 + x_2 \geq 1$$

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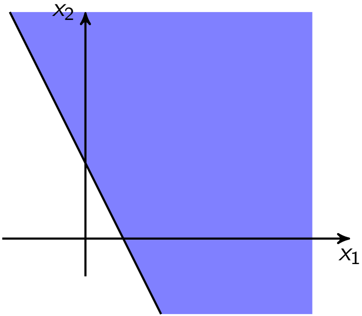

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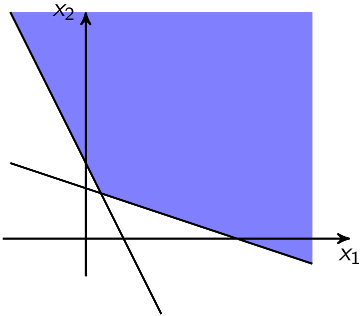
$$\begin{aligned} 2x_1 + x_2 &\geq 1 \\ x_1 + 3x_2 &\geq 2 \end{aligned}$$



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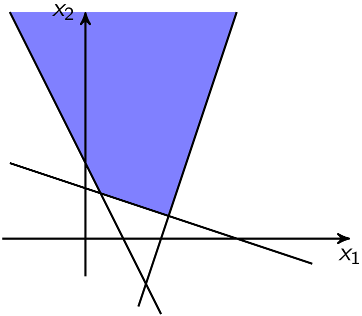
$$\begin{aligned} 2x_1 + x_2 &\geq 1 \\ x_1 + 3x_2 &\geq 2 \\ -3x_1 + x_2 &\geq -3 \end{aligned}$$



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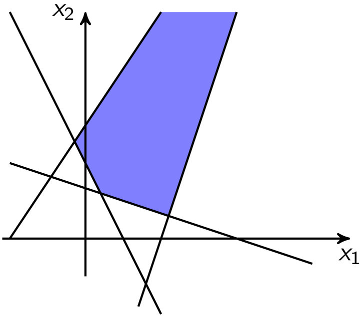

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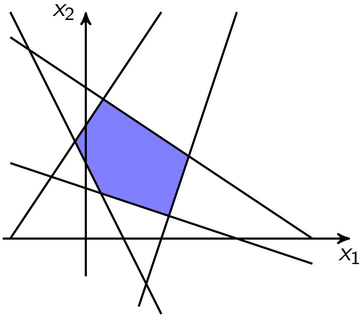
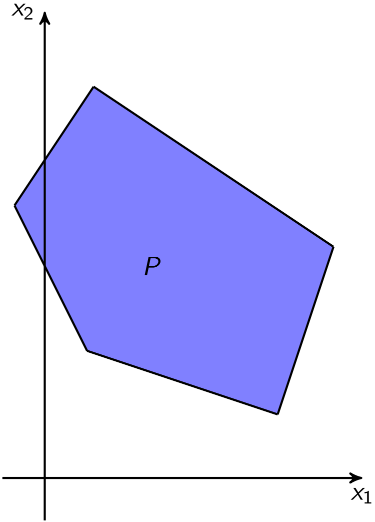


Observe the corners !

Solving a linear program

Example:

$$\begin{cases} \min \ x_1 \\ x \in P \end{cases}$$



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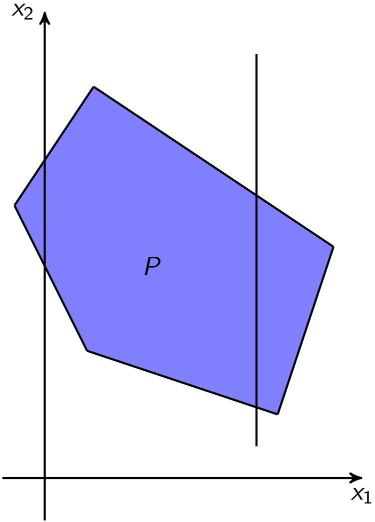

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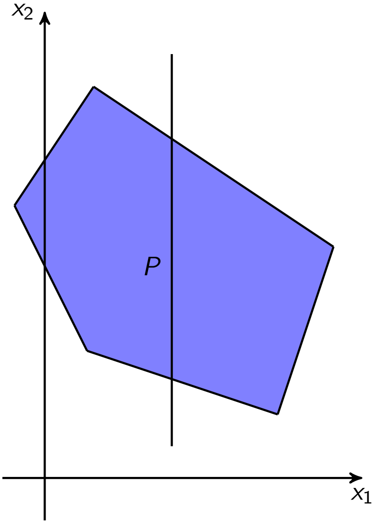

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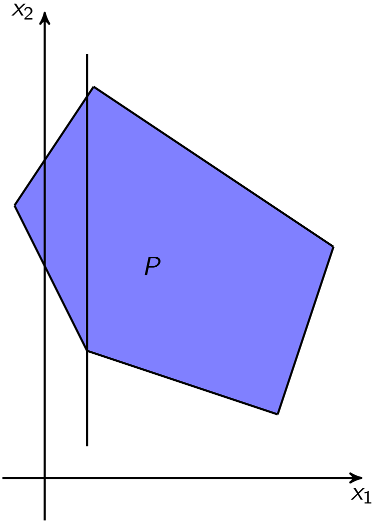

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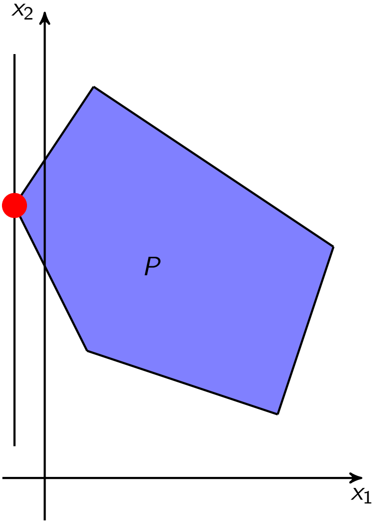

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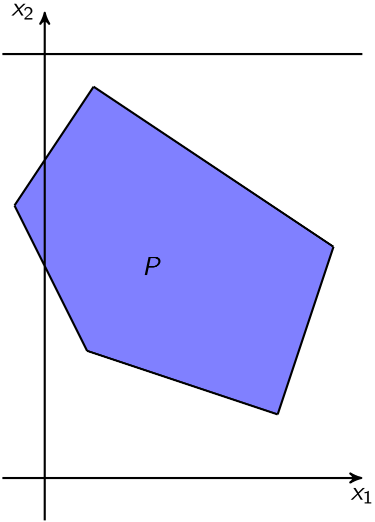

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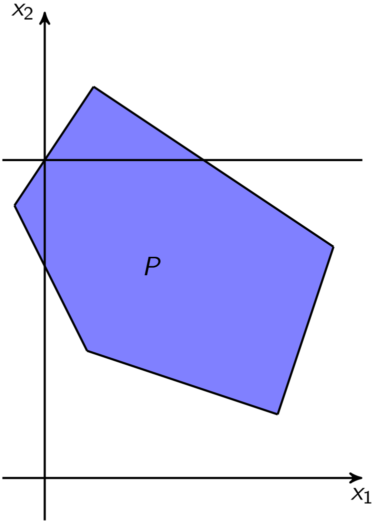

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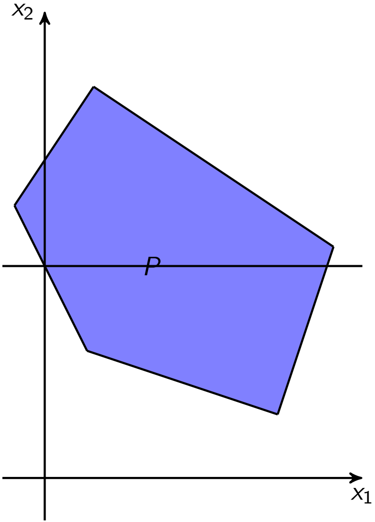

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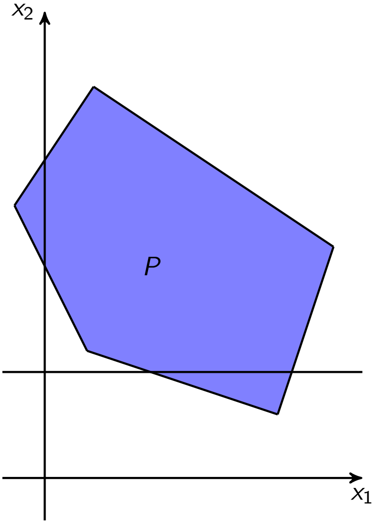

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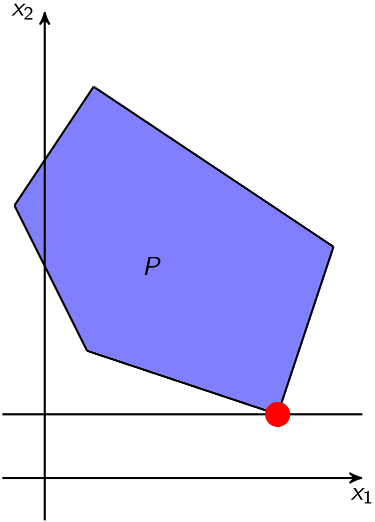

Solving a linear program

Example:

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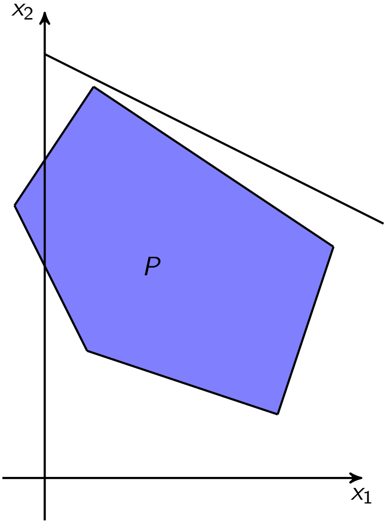

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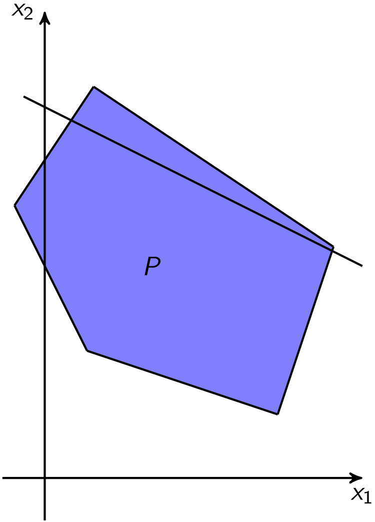

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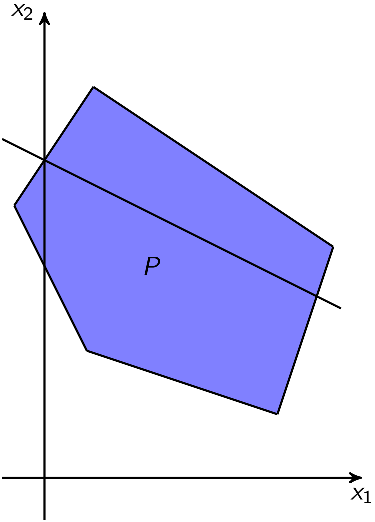

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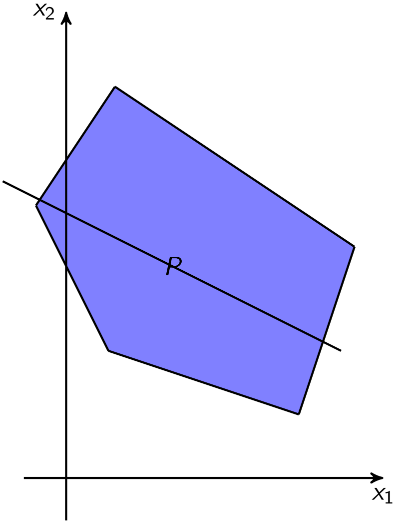

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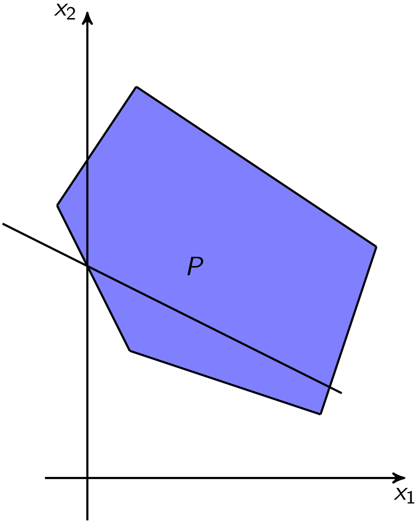

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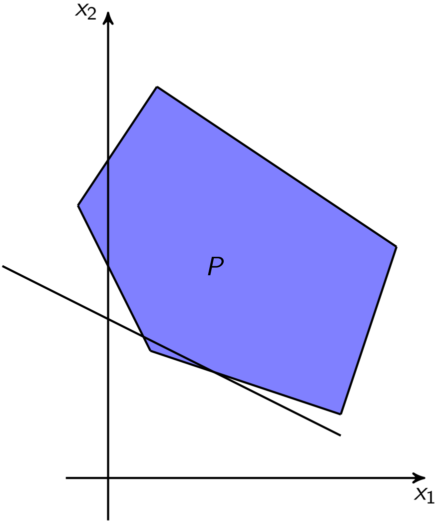

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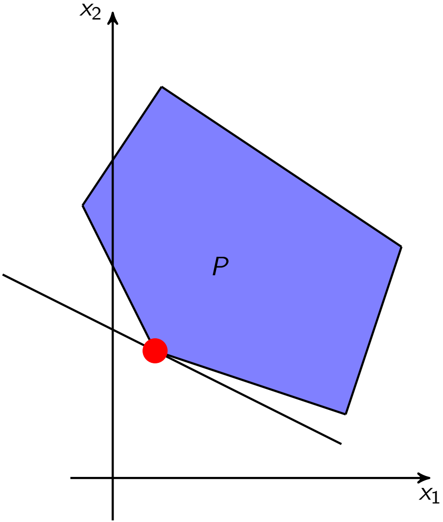

Basic Feasible Solutions (BFS)

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a nonempty and bounded polyhedron (this requires more inequalitites than variables) and consider the linear program

$$\begin{cases} \min c^t x \\ Ax \leq b \end{cases}$$

A *basic solution (BS)* of the system $Ax \leq b$ is any point obtained by forcing n of these inequalities to be satisfied as equalities, and such that the resulting system of linear equations has a unique solution (i.e. n linearly independent equations).

If the resulting point lies in P (i.e. if it satisfies the remaining inequalities) it is called a *basic feasible solution (BFS)*.

Theorem

- *A polytope* $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ *has finitely many BS.*
- $\min\{c^t x : x \in P\}$ *is attained at least at one BFS.*

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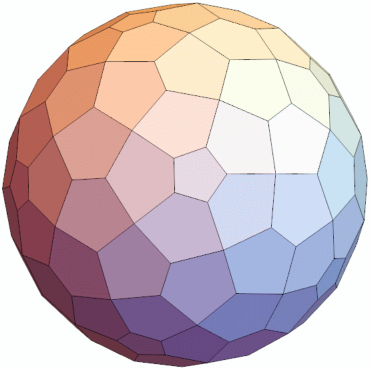
The Simplex method – idea

A brute force algorithm for solving a linear program is to try out all the possible BS until we find the best BFS (if any). This is prohibitively slow.

The Simplex method explores the BFS in a smarter way, moving from one BFS to a "neighboring" one which has strictly smaller objective value. Two BFS are neighbors if the equalities that define each one differ only by a single equation.



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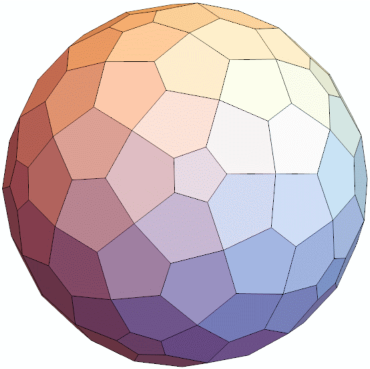

Simplex method – properties

While the Simplex can be very slow in the worst case, it is extremely efficient in most practical cases.

The Simplex method allows for unbounded polyhedra and is able to detect when the problem is infeasible, when it has no optimal solutions, and when it does. In the latter case it finds one optimal BFS.

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Simplex method – Degeneracy & Cycling