Game theory interacting with philosophy, psychology and concrete problems

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What is Game Theory?

In
Pragmatics of Human Communication

Paul Watzlawick, Janet Helmick Beavin and Don D. Jackson

state the axioms at the basis of the human communication.

The first axiom is

Impossibility of avoiding communication

A communication unit is a message, a series of messages among people constitutes an interaction.

Game theory is the mathematics of interaction.

The definition of Game

A Game is a situation where

1) there are players, usually more than one;
2) there is an initial situation;
3) there are well defined rules of behavior for the players;
4) there are final situations, or outcomes, leading to the end of the game;
5) the players have preferences on the outcomes.

The axioms of Game Theory

The basic assumptions:

Players are:

1) selfish;
2) rational.

Selfishness

Players do care only about their personal preferences.

Other's players preferences are taken into account only in order to maximize the own personal satisfaction.

Rationality

Several levels of rationality.

The basic one:

To be able to express coherent preferences over the outcomes of the game.

Mathematically:

To be able to provide a complete, reflexive, transitive relation over the outcomes of the game.

And more:

When necessary to be able to provide a utility function for the preference relation.

More sophisticated levels of rationality

Players are able to:

1) make an unlimited analysis on the consequences of everybody actions;

2) use the probability laws in presence of aleatory events;

3) use the theory of decisions whenever possible.

Unlimited analysis on the consequences of everybody actions

Consider the following situation:

1) Everybody writes an integer n with $1 \leq n \leq 100$;

2) the mean of the answer is calculated;

3) those writing the integer closer to 95% of the mean win an interesting prize.

The beauty contest

The former is a way to mathematically describe the following celebrated sentence, related to a special contest[1], and due to the Economist J. Keynes:

It is not a case of choosing those [faces] which, to the best of one's judgement, are really the prettiest, nor even those which average opinion genuinely thinks the prettiest. We have reached the third degree where we devote our intelligencies to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practice the fourth, fifth and higher degrees.[2]

[1] see f.i. http://missrheingold.com/history/.

[2] J. Keynes, General theory of Employment, Interest and Money. Cambridge University Press, 1936.

Figure from *Scacchi e scimpanzé*[a]

[a] R.Lucchetti, Bruno Mondadori Editore, 2012

Use of the probability laws

Lotteries

In the first shop:

Alternative A:

gain	probability
2500	33%
2400	66%
0	1%

Alternative B:

gain	probability
2500	0%
2400	100%
0	0%

Use of the probability laws

In the second shop:

Alternative C:

gain	probability
2500	33%
0	67%

Alternative D:

gain	probability
2400	34%
0	66%

Allais[a] experiment

[a] Nobel Laureate in Economics, 1988

According to probability, the agent should evaluate, in the first shop, what is better between

$$0.33u(2500) + 0.66u(2400)$$

and

$$u(2400)$$

The problem in the second shop is absolutely the same!

However in a sample of (well educated) 72 people, in the first shop 82% of them decided to play the Lottery B, in the second shop, 83% of the people selected lottery C.

Use of the theory of decisions

Basic assumption in Decision theory: The decision maker does not take a decision z if she has available a decision x such that the final outcome obtained with x is strictly preferable for her than the outcome obtained with z.

Mathematically, the decision maker maximizes one among her utility functions.

A first consequence

Obvious extension to interactive decision theory:

The decision maker does not take a decision z if she has available a decision x such that the final outcome with x is strictly preferable for her than the outcome obtained with z, for every possible combination of actions of the other players.

Mathematically, z is not taken if there exists x such that

$$f(x,y) > f(z,y) \qquad \forall y$$ [3]

[3] f is the utility function of the player, y represents any possible combination of the actions of all other players.

First consequence of the axioms

$$\begin{pmatrix} (10,10) & (3,15) \\ (15,3) & (5,5) \end{pmatrix}$$

The most elegant way to efficiently summarize one of the most influential books in the social philosophy.

In such condition there is no place for industry, because the fruit thereof is uncertain, and consequently, not culture of the earth, no navigation, nor the use of commodities that may be imported by sea, no commodious building, no instruments of moving and removing such things as require much force, no knowledge of the face of the earth, no account of time, no arts, no letters, no society, and which is worst of all, continual fear and danger of violent death, and the life of man, solitary, poor, nasty, brutish, and short.

T. Hobbes, Leviathan 1651

And more

Consider the two games:

$$\begin{pmatrix} (10,10) & (3,15) \\ (15,3) & (5,5) \end{pmatrix},$$

$$\begin{pmatrix} (8,8) & (2,7) \\ (7,2) & (0,0) \end{pmatrix}.$$

Observe, in the second game, in any outcome, both players get more than in the corresponding outcome in the second one. However:

Better to choose the worst scenario!

And more!

Consider the two games:

$$\begin{pmatrix} (10,10) & (3,5) \\ (5,3) & (1,1) \end{pmatrix}$$

$$\begin{pmatrix} (1,1) & (11,0) & (4,0) \\ (0,11) & (10,10) & (3,5) \\ (0,4) & (5,3) & (1,1) \end{pmatrix}.$$

Observe, in the second game the players have available the same choices and outcomes, plus further outcomes. But:

Having less options is better than having more options!

Basics on coalitional games

A coalitional game is a pair (N, v), N is a finite set (usually players) and $v : 2^N \mapsto \mathbb{R}$ is the characteristic function, with $v(\emptyset) = 0$.
G^N is the set all games with N the set of players ($N = \{1, 2, \dots, n\}$).

An example

$[q, w_1, \dots, w_n]$ is the following N-players game:

$$v(T) = \begin{cases} 1 & \text{if } w(T) := \sum_{i \in T} w_i \geq q \\ 0 & \text{otherwise} \end{cases}.$$

Two more examples

- There are one seller and two potential buyers for a good. The seller (Player 1) evaluates the good a. Players two and three evaluate it b and c, respectively, with $a < b < c$. The associated game:

$$v(\{1\}) = a, v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0,$$

$$v(\{1,2\}) = b, v(\{1,3\}) = c, \ v(N) = c.$$

- There are two sellers and one potential buyer for a good. The associated game:

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,2\}) = 0,$$

$$v(\{1,3\}) = v(\{2,3\}) = v(N) = 1.$$

Semivalues

A semivalue (Carreras and Freixas 1999; 2000) converts information about the worth coalitions achieve into a personal attribution to each of the players:

$$\pi_i^{\mathbf{p}}(v) = \sum_{S \subset N : i \notin S} p_s\big(v(S \cup \{i\}) - v(S)\big)$$

for each $i \in N$; p_s represents the probability that $i \notin S$ joins coalition $S \in 2^N$ (of cardinality s).

$$\sum_{s=0}^{n-1} \binom{n-1}{s} p_s = 1.$$

$p_s > 0$ for all s is regularity of the semivalue.

The most famous semivalues

- The Shapley value (Shapley 1953) $\pi^{\hat{\mathbf{p}}}(v)$:

$$\hat{p}_s = \frac{1}{n\binom{n-1}{s}} = \frac{s!(n-s-1)!}{n!}$$

for each $s = 0, 1, \ldots, n-1$.

- The Banzhaf power index (Banzhaf III 1964) $\pi^{\tilde{\mathbf{p}}}(v)$:

$$\tilde{p}_s = \frac{1}{2^{n-1}}$$

for each $s = 0, 1, \ldots, n-1$.

- The binomial semivalues $p_s = p^s(1-p)^{n-s-1}$, for $0 < p < 1$
- The segment semivalues (like $[S, B]$).

From the papers

Lucchetti R., Moretti S., Patrone F. and Radrizzani P. The Shapley and Banzhaf indices in microarray games. Computers and Operations Research (2009).

Lucchetti R. and Radrizzani P. Data Analysis Via Weighted Indices and Weighted Majority Games. Computational Intelligent Methods for Bioinformatics and Biostatistics II, Masulli, Peterson, Tagliaferri (Eds), Lecture Notes in Computer Science, Springer (2010)

Lucchetti, R., Radrizzani,P. and Munarini, E. A new family of regular semivalues and applications. Int. J. Game Theory 40,(2011), p. 655-675

A $(n \times m)$ matrix $M = (m_{ij})$ is given, such that m_{ij} is either zero or one. Moreover for every j there is i with $m_{ij} = 1$.

Given the column $m_{.j}$, $j = 1, \dots, m$, call *support* the set supp $m_{.j} = \{i : m_{ij} = 1\}$; consider the associated unanimity game v^j generated by supp $m_{.j}$:

$$u_{supp\ m_{.j}}(R) = \begin{cases} 1 & \text{if supp } m_{.j} \subseteq R \\ 0 & \text{otherwise} \end{cases}.$$

The microarray game associated to $M = (m_{ij})$ is

$$v = \frac{1}{m} \sum_{j=1}^{m} v^j.$$

The meaning

gene1	0.5	0.2	1
gene2	0.4	1	0.3
gene3	0.8	0.4	0.2

gene1	0.7	0.3	1.8	0.8
gene2	0.1	0.2	0.5	0.9
gene3	1	0.6	1.7	0.1

gene1	0	0	1	0
gene2	1	1	0	0
gene3	1	0	1	1

$$v(1) = 0,\ v(2) = v(3) = v(1,2) = \frac{1}{4},\ v(1,3) = \frac{1}{2},\ v(2,3) = \frac{3}{4},\ v(N) = 1.$$

Semivalues on unanimity games

On the unanimity game

$$u_T(R) = \begin{cases} 1 & \text{if } T \subseteq R \\ 0 & \text{otherwise} \end{cases}.$$

To the players in T:

Shapley

$$\frac{1}{t}$$

Banzhaf

$$\frac{1}{2^{t-1}}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Gene 1 is the last ranked for Shapley

Gene 1 is the first ranked for Banzhaf

A new family of regular semivalues

Let $a \in \mathbb{R}$, $a > 0$.

Definition

Let σ^a, be defined on unanimity game u_R:

$$\sigma_i^a(u_R) = \begin{cases} \frac{1}{r^a} & \text{if } i \in R \\ 0 & \text{otherwise} \end{cases},$$

extended by linearity on G^N.

Shapley:a=1

The main result

Theorem

σ^a is a regular semivalue for all $a > 0$. The 2-value fulfills:

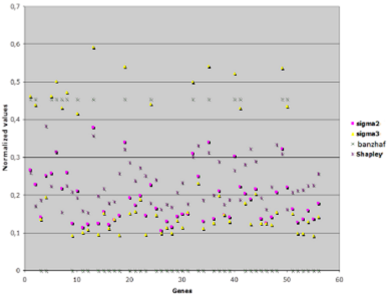
$$\sigma_i^2(v) = \sum_{S \subseteq 2^{N\setminus\{i\}}} \left(\frac{s!(n-1-s)!}{n!} \sum_{k=s+1}^{n} \frac{1}{k} \right) m_i(S). \tag{1}$$

Corollary

The family of the weighting coefficients of the values σ^a, $a \in \mathbb{R}_+$, is an open curve in the simplex of the regular semivalues. Addition of the Banzhaf value provides a one-point compactification of the curve.

A first experiment

Microarray game defined on a tumor/normal data set microarray.princeton.edu/oncology/affydata/index.html containing the expression levels of a set of 2000 genes measured using Affymetrix oligonucleotide microarray for a set of 40 tumor samples, 22 normal samples



An extension of the model

For each gene i,

1) we consider the reference interval $I_i = [m_i, M_i]$,
2) we calculate the standard deviation s_i relative to the data of the gene (with respect to the data taken from the reference group),
3) we set $N_i^k = [m_i - ks_i, M_i + ks_i]$, $k = 1, \dots$
4) we assign value k if gene falls in the set $N_i^k \setminus N_i^{k-1}$.

The Generalized microarray matrix $\hat{M}$, has non negative integers entries.

Next, we select the first 50 genes ranked (by the Shapley index) with the generalized microarray game.

We consider, for each column j, the weighted majority game such that $w_{ij} = \hat{m}_{ij}$.

Finally we rank the genes with Shapley, Banzhaf, σ^2.

A second experiment:data concerning early onset colon rectal cancer

Gene expression analysis performed by using Human Genome U133A-Plus 2.0 GeneChip arrays (Affymetrix, Inc.,).
Data set containing 10 healthy samples and 12 derived from tumor tissues.

7 genes were already identified as a potentially prediction of early onset colorectal cancer: CYR61, FOS, FOSB, EGR1, VIP, KRT24, UCHL1.

	B	S	σ^2
FOSB	2	1	1
CYR61	1	2	2
FOS	3	3	3
VIP	5	5	6
EGR1	10	9	9
KRT24	45	35	35

The problem

To provide a sane kidney to patients suffering of severe kidney disease.

Two methods:

1) receiving a kidney from a cadaver;
2) receiving a kidney form a living donor.

Some data

The italian situation:

1) 611 transplants in 1992;
2) 1533 in 2009.

The donations from cadavers do not suffice.

Medical problems related to a donation

1) blood incompatibility;
2) tissue incompatibility.

The first ones are related to the blood group of the donor and the patient $(0, A, B, AB)$.

The second ones are related to the so called HLA (human leukocyte antigens).

Blood incompatibility: You *cannot* receive a protein (A, B) you do not have in your blood.
Tissue incompatibility expensive and delicate test, to be repeated: incompatibilities can arise any time.

The main assumptions

1) Patients have preferences over the available kidneys;
2) they can decide to enroll to a waiting list;
3) usually, but not necessarily, they have a living donor.

Other assumptions

Different models rely on different assumptions on the possible exchanges:

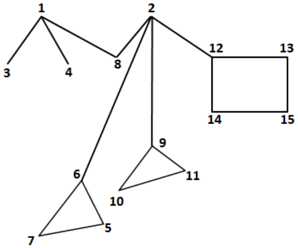
1) pairwise exchange;
2) multiple exchange with no waiting list;
3) multiple exchange including waiting list.

The only possibility is that patient A receives the kidney of the donor B and conversely.

Patient A_1 receives the kidney of the donor of A_2, ... patient A_{n-1} receives the kidney of the donor of A_n, A_n receives the kidney of the donor of A_1.

Patient A_1 receives the kidney of the donor of A_2, ... patient A_{n-1} receives the kidney of the donor of A_n, the donor of A_1 donates the kidney to the waiting list, patient A_n gets high priority on the waiting list.

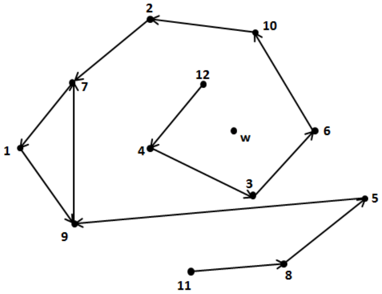
An example for the pairwise



In this case preferences are dichotomous: every compatible kidney is equally acceptable and strictly better than any incompatible.

The case with the waiting list

Case with several variants:



Here preferences are strict. Cycles are formed and taken out from the process.

Chains are considered. Here several variants are possible.

Game theoretical aspects

Contributions of game theory to this area:

1) algorithms for the possible matching;
2) efficiency of the matching;
3) priority mechanisms among equally efficient matching;
4) manipulability (on declaring preferences).

A problem in Social Choice

Based on two papers: Stefano Moretti and Alexis Tsoukiàs Ranking sets of possibly interacting objects using Shapley extensions Proceedings of the 13th International Conference on Principles of Knowledge Representation and Reasoning (KR2012), June 10-14, 2012, Rome.

Roberto Lucchetti, Stefano Moretti and Fioravante Patrone Ranking sets of interacting objects via semivalues Missing!.

Central question(s)

How to derive a ranking over the set of all subsets of a finite set N "compatible" with a given ranking over the elements of N?

- Most papers dealing with this issue provide an axiomatic approach (Kannai and Peleg (1984), Barbera et al (2004), Bossert (1995), Fishburn (1992), Roth (1985) etc.)

- **Extension axiom**: given total preorder $\succcurlyeq$ on N, a total preorder $\sqsupseteq$ on 2^N is an *extension* of $\succcurlyeq$ if for each $x, y \in N$,

$$\{x\} \sqsupseteq \{y\} \Leftrightarrow x \succcurlyeq y$$

Well-known extensions prevent interaction

Axiom (Responsiveness, RESP)

A total preorder $\sqsupseteq$ *on* 2^N *satisfies the* responsiveness *property if for all* $S \in 2^N$ *such that* $i, j \notin S$

$$(S \cup \{i\}) \sqsupseteq (S \cup \{j\}) \Leftrightarrow \{i\} \sqsupseteq \{j\}$$

Acceptable preorders

Every (normalized) utility function associated to a total preorder $\sqsupseteq$ on 2^N originates a TU-Game.

Let $V(\sqsupseteq)$ be the class of coalitional games representing $\sqsupseteq$.

Then we consider a prorder $\sqsupseteq$ on 2^N *acceptable*, according to the semivalue p if for each $v \in V(\sqsupseteq)$
the ranking provided by the semivalue p respects the initial ranking over the objects.

Observe, in particular the ranking is independent from the chosen utility function.

A characterization

Via a simple formula, setting $x_s = p_s + p_{s+1}$ and

$$\left[\sum_{S: i,j \notin S, |S|=s} [v(S \cup \{i\}) - v(S \cup \{j\}] \right] = a_{s+1}^{ijv},$$

Semivalues acceptable for $\sqsupseteq$ can be found by solving the following semi-infinite system of linear inequalities:

$$\begin{aligned} a_1^{ijv} x_1 + a_2^{ijv} x_2 + \cdots + a_{n-1}^{ijv} x_{n-1} \geq 0, \quad v \in V(\sqsupseteq), \quad i \sqsupseteq j, \\ x_1 \geq 0, \ldots \quad x_{n-1} \geq 0 \quad x \neq 0 \end{aligned}$$

The results obtained

1) To reduce the semi-infinite problem to a finite and handable one, for every possible semivalue;
2) To *characterize* with a simple condition those preorders that are acceptable for every semivalue.