Semivalues and applications

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The microarray technique

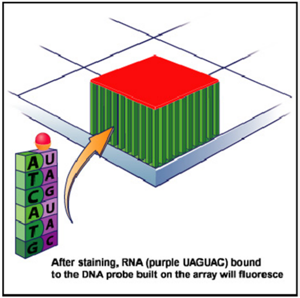
It is used to check whether the DNA of an individual contains genes which are under/over expressed (i.e. abnormally expressed)

How does it work?

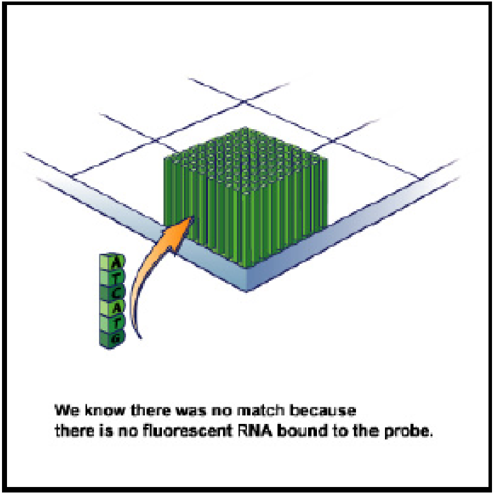

AFFYMETRIX
GeneChip®
Human Genome
U133 Plus 2.0
P/N: 520019
Lot #: SAMPLES
Exp. Date: 00/00/00
For Research Use Only

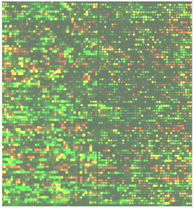
Adenine,Guanine,Thymine,Cytosine are nucleobases of the DNA/RNA

DNA	$\hookrightarrow$	RNA
ADENINE	$\hookrightarrow$	THYMINE
GUANINE	$\hookrightarrow$	CYTOSINE
THYMINE/URACILE	$\hookrightarrow$	ADENINE
CYTOSINE	$\hookrightarrow$	GUANINE

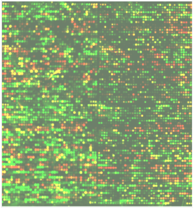


If the gene is highly expressed,many RNA molecules will stick to the probe and the probe location will shine brightly when the laser hit it.

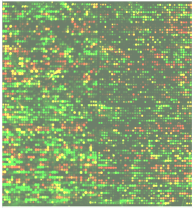

A
T
C
A
T
G
We know there was no match because
there is no fluorescent RNA bound to the probe.



Array1



Array2



Array3

...

	array 1	array 2	array 3	array 4	...
gene 1	0,67	0,45	1,32	1,34	...
gene 2	1,01	1,13	1,54	2,13	...
gene 3	1,38	1,21	1,23	0,12	...
gene 4	0,65	0,98	0,54	...	...

Basics on coalitional games

A coalitional game is a pair (N, v), N is a finite set (usually players) and $v : 2^N \mapsto \mathbb{R}$ is the characteristic function, with $v(\emptyset) = 0$.
G^N is the set all games with N the set of players ($N = \{1, 2, \dots, n\}$)

Observe:

$$G^N \simeq \mathbb{R}^{2^n - 1}.$$

A base for G^N is the following family of unanimity games:

(N, u_R), $\emptyset \neq R \subseteq N$:

$$u_R(T) = \begin{cases} 1 & \text{if } R \subseteq T \\ 0 & \text{otherwise} \end{cases}.$$

Some examples

- There are one seller and two potential buyers for a good. The seller (Player 1) evaluates the good a. Players two and three evaluate it b and c, respectively, with $a < b < c$ The associated game:

$$v(\{1\}) = a, v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0,$$

$$v(\{1,2\}) = b, v(\{1,3\}) = c, \; v(N) = c.$$

- There are two sellers and one potential buyer for a good. The associated game:

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,2\}) = 0,$$

$$v(\{1,3\}) = v(\{2,3\}) = v(N) = 1.$$

A further one

$[q, w_1, \dots, w_n]$ is the following N-players game:

$$v(T) = \begin{cases} 1 & \text{if } w(T) := \sum_{i \in T} w_i \geq q \\ 0 & \text{otherwise} \end{cases}.$$

Semivalues

A semivalue (Carreras and Freixas 1999; 2000) converts information about the worth coalitions achieve into a personal attribution to each of the players:

$$\pi_i^{\mathbf{p}}(v) = \sum_{S \subset N : i \notin S} p_s \big(\, v(S \cup \{i\}) - v(S) \big)$$

for each $i \in N$; p_s represents the probability that $i \notin S$ joins coalition $S \in 2^N$ (of cardinality s).

$$\sum_{s=0}^{n-1} \binom{n-1}{s} p_s = 1$$

$p_s > 0$ for all s is regularity of the semivalue.

The most famous semivalues

- The Shapley value (Shapley 1953) $\pi^{\hat{\mathbf{p}}}(v)$:

$$\hat{p}_s = \frac{1}{n\binom{n-1}{s}} = \frac{s!(n-s-1)!}{n!}$$

for each $s = 0, 1, \dots, n-1$.

- The Banzhaf power index (Banzhaf III 1964) $\pi^{\tilde{\mathbf{p}}}(v)$:

$$\tilde{p}_s = \frac{1}{2^{n-1}}$$

for each $s = 0, 1, \dots, n-1$.

binomial semivalues $p_s = p^s(1-p)^{n-s-1}$, for $0 < p < 1$

Segment semivalues (like $[S, B]$).

From the papers

Lucchetti R., Moretti S., Patrone F. and Radrizzani P. The Shapley and Banzhaf indices in microarray games. Computers and Operations Research (2009).

Lucchetti-P. Radrizzani Data Analysis Via Weighted Indices and Weighted Majority Games Computational Intelligent Methods for Bioinformatics and Biostatistics II, Masulli, Peterson, Tagliaferri (Eds), Lecture Notes in Computer Science, Springer (2010) p.179-190.

R. Lucchetti-P. Radrizzani, E. Munarini A new family of regular semivalues and applications Int. J. Game Theory 40,(2011), p. 655-675

A $(n \times m)$ matrix $M = (m_{ij})$ is given, such that m_{ij} is either zero or one. Moreover for every j there is i with $m_{ij} = 1$.

Given the column $m_{.j}$, $j = 1, \dots, m$, call *support* the set supp $m_{.j} = \{i : m_{ij} = 1\}$; consider the associated unanimity game v^j generated by supp $m_{.j}$

The microarray game associated to $M = (m_{ij})$ is

$$v = \frac{1}{m} \sum_{j=1}^{m} v^j.$$

The meaning

gene1	0.5	0.2	1
gene2	0.4	1	0.3
gene3	0.8	0.4	0.2

gene1	0.7	0.3	1.8	0.8
gene2	0.1	0.2	0.5	0.9
gene3	1	0.6	1.7	0.1

gene1	0	0	1	0
gene2	1	1	0	0
gene3	1	0	1	1

$$v(1) = 0,\ v(2) = v(3) = v(1,2) = \frac{1}{4},\ v(1,3) = \frac{1}{2},\ v(2,3) = \frac{3}{4},\ v(N) = 1.$$

Axioms

Axiom

Let $v \in \mathcal{G}^N$. The solution ψ has the dummy player (DP) property, if for each player $i \in N$ such that $v(A \cup \{i\}) = v(A) + v(\{i\})$ for all $A \subset N \setminus \{i\}$, then

$$\psi_i(v) = v(\{i\}). \tag{1}$$

Axiom

Let be given a finite set N of players, and let $\pi : N \to N$ be a permutation of N. Given the game v, denote by $\pi^* v$ the following game: $(\pi^* v)(A) = v(\pi(A))$, and by $\pi^*(x) = (x_{\pi(1)}, \dots, x_{\pi(n)})$. The solution ψ has the symmetry (S) property if, for each permutation π on N, $\psi(\pi^* v) = \pi^*(\psi(v))$.

Semivalues on unanimity games

On the unanimity game u_T, to the non null players:

Shapley

$$\frac{1}{t}$$

Banzhaf

$$\frac{1}{2^{t-1}}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Gene 1 is the last ranked for Shapley

Gene 1 is the first ranked for Banzhaf

The definition

Let $a \in \mathbb{R}$, $a > 0$.

Definition

Let σ^a, be defined on unanimity game u_R:

$$\sigma_i^a(u_R) = \begin{cases} \frac{1}{r^a} & \text{if } i \in R \\ 0 & \text{otherwise} \end{cases},$$

extended by linearity on G^N.

Shapley:a=1

A first theorem

Theorem

There exists one and only one value ϕ fulfilling the symmetry, linearity and dummy player properties, and assigning a_s to all non null players in the unanimity game u_S, for all coalitions S such that $|S| = s$, where $a_1 = 1$ and $a_s > 0$ for $s = 2, \dots, n$. Moreover ϕ fulfills the formula:

$$\phi_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} \left(\sum_{k=0}^{n-s-1} \binom{n-s-1}{k} (-1)^k a_{s+k+1} \right) m_i(S). \tag{2}$$

The main result

Theorem

σ^a is a regular semivalue for all $a > 0$. The 2-value fulfills:

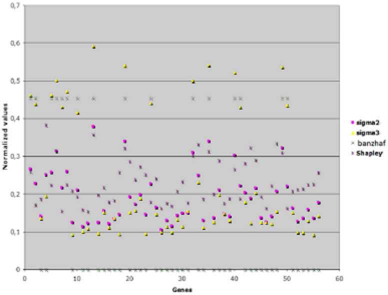
$$\sigma_i^2(v) = \sum_{S \subseteq 2^{N \setminus \{i\}}} \left(\frac{s!(n-1-s)!}{n!} \sum_{k=s+1}^{n} \frac{1}{k} \right) m_i(S). \tag{3}$$

Corollary

The family of the weighting coefficients of the values σ^a, $a \in \mathbb{R}_+$, is an open curve in the simplex of the regular semivalues. Addition of the Banzhaf value provides a one-point compactification of the curve.

A first experiment

Microarray game defined on a tumor/normal data set microarray.princeton.edu/oncology/affydata/index.html containing the expression levels of a set of 2000 genes measured using Affymetrix oligonucleotide microarray for a set of 40 tumor samples, 22 normal samples



An extension of the model

To each gene i, we consider the reference interval $I_i = [m_i, M_i]$,
the standard deviation s_i relative to the data of the gene (with respect to the data taken from the reference group),
set $N_i^k = [m_i - ks_i, M_i + ks_i]$, $k = 1, \dots$
assign value k if gene falls in the set $N_i^k \setminus N_i^{k-1}$.

Generalized microarray matrix $\hat{M}$, having non negative integers entries.

Select the first 50 genes ranked with a standard microarray game

Consider, for each column j the weighted majority game such that $w_{ij} = \hat{m}_{ij}$

Find the ranking given by the indices Shapley, Banzhaf, σ^2

A second experiment:data concerning early onset colon rectal cancer

Gene expression analysis performed by using Human Genome U133A-Plus 2.0 GeneChip arrays (Affymetrix, Inc.,).
Data set containing 10 healthy samples and 12 derived from tumor tissues.

7 genes were already identified as a potentially prediction of early onset colorectal cancer: CYR61, FOS, FOSB, EGR1, VIP, KRT24, UCHL1.

	B	S	σ^2
FOSB	2	1	1
CYR61	1	2	2
FOS	3	3	3
VIP	5	5	6
EGR1	10	9	9
KRT24	45	35	35

Based on two papers

Stefano Moretti and Alexis Tsoukiàs Ranking sets of possibly interacting objects using Shapley extensions Proceedings of the 13th International Conference on Principles of Knowledge Representation and Reasoning (KR2012), June 10-14, 2012, Rome.

Roberto Lucchetti, Stefano Moretti and Fioravante Patrone Ranking sets of interacting objects via semivalues Submitted.

Central question(s)

How to derive a ranking over the set of all subsets of a finite set N "compatible" with a given ranking over the elements of N?

- Most papers dealing with this issue provide an axiomatic approach (Kannai and Peleg (1984), Barbera et al (2004), Bossert (1995), Fishburn (1992), Roth (1985) etc.)

- **Extension axiom**: given total preorder $\succcurlyeq$ on N, a total preorder $\sqsupseteq$ on 2^N is an *extension* of $\succcurlyeq$ if for each $x, y \in N$,

$$\{x\} \sqsupseteq \{y\} \Leftrightarrow x \succcurlyeq y$$

Example: max and min

The simplest extensions are the *max* and the *min* extensions.

- For instance, let $N = \{1, 2, 3\}$ and $1 \succ 2 \succ 3$.
According to the max extension, for each $S, T \in 2^N \setminus \{\emptyset\}$, we have

$$(S \sqsupseteq^{\max} T) \Leftrightarrow (\text{ best}(S) \succcurlyeq \text{ best}(T))$$

So the extension $\sqsupseteq^{\max}$ of $\succcurlyeq$ is:
$\{1, 2, 3\} \backsimeq^{\max} \{1, 3\} \backsimeq^{\max} \{1, 2\} \backsimeq^{\max} \{1\} \sqsupset^{\max} \{2\} \backsimeq^{\max} \{2, 3\} \sqsupset^{\max} \{3\}$

Interaction is important

- Most of the axiomatic approaches from the literature use axioms aimed at preventing interaction among the objects in N.

- However many decision problems require learning preferences over sets of possibly interacting objects.

- For example, you do not need to be Blanc to understand that taking the eleven first ranked players on L'Équipe does not make PSG win the Champion's League!

- or picking the top k items ranked by google does not always yield the optimal subset for building a music playlist.

Which kind of interaction effects?

- Let $N = \{x, y, z\}$ and suppose that an agent's preference is such that $x \succcurlyeq y$, $x \succcurlyeq z$ and $y \succcurlyeq z$.

- Trying to extend $\succcurlyeq$ to 2^N, one could guess that set $\{x, y\}$ is better than $\{y, z\}$, because the agent will receive both y and x instead of y and z (and x is preferred to z).

- However, in case of incompatibility among x and y, or complementarity effects between y and z, the relative ranking between the two sets $\{x, y\}$ and $\{y, z\}$ could be reversed.

Well-known extensions prevent interaction

Axiom (Responsiveness, RESP)

A total preorder $\sqsupseteq$ *on* 2^N *satisfies the* responsiveness *property if for all* $S \in 2^N$ *such that* $i, j \notin S$

$$(S \cup \{i\}) \sqsupseteq (S \cup \{j\}) \Leftrightarrow \{i\} \sqsupseteq \{j\}$$

- This axiom was introduced by Roth (1985) studying colleges' preferences for the "college admissions problem" (see also Gale and Shapley (1962)).

- Bossert (1995) used the same property for ranking sets of alternatives with a fixed cardinality and to characterize the class of *rank-ordered lexicographic* extensions.

Well known extensions

The following extensions satisfy the RESP property:

- max and min extensions (Barberà, Bossert, and Pattanaik 2004)

- lexi-min and lexi-max extensions (Barberà, Bossert, and Pattanaik 2004)

- median-based extensions (Nitzan and Pattanaik 1984)

- rank-ordered lexicographic extensions (Bossert 1995)

- many others...

p-aligned total preorders

Key (and simple) remark Every (normalized) utility function associated to a total preorder $\sqsupseteq$ on 2^N originates a TU-Game.

Let $V(\sqsupseteq)$ be the class of coalitional games representing $\sqsupseteq$

DEF. Let π^p be a semivalue. A total prorder $\sqsupseteq$ on 2^N is **p**-*aligned* if for each $v \in V(\sqsupseteq)$

$$\{i\} \sqsupseteq \{j\} \Leftrightarrow \pi_i^{\hat{p}}(v) \geq \pi_j^p(v)$$

for all $i, j \in N$. ■.

In other words, the ranking assigned by the semivalue to the players (objects) respects the initial ranking and does *not* depend from the utility function selected to represent the ordering on 2^N.

A characterization

Setting $x_s = p_s + p_{s+1}$ and

$$\left[\sum_{S: i,j \notin S, |S|=s} [v(S \cup \{i\}) - v(S \cup \{j\}] \right] = a_{s+1}^{ijv},$$

Semivalues $\pi^{\mathbf{p}}$ aligned with $\sqsupseteq$ can be found by solving the semi-infinite system of linear inequalities:

$$a_1^{ijv} x_1 + a_2^{ijv} x_2 + \cdots + a_{n-1}^{ijv} x_{n-1} \geq 0, \quad v \in V(\sqsupseteq), \quad i \sqsupseteq j,$$
$$x_1 \geq 0, \ldots \quad x_{n-1} \geq 0 \quad x \neq 0$$

Example: a Shapley-aligned total preorder

For each coalitional game v, the Shapley value is denoted by $\phi(v) = \pi^{\hat{p}}(v)$.
Let $N = \{1, 2, 3\}$ and let $\sqsupseteq^a$ be a total preorder on N such that $\{1, 2, 3\} \sqsupset^a \{3\} \sqsupset^a \{2\} \sqsupset^a \{1, 3\} \sqsupset^a \{2, 3\} \sqsupset^a \{1\} \sqsupset^a \{1, 2\} \sqsupset^a \emptyset$.

For every $v \in V(\sqsupseteq^a)$

$$\phi_2(v) - \phi_1(v) = \frac{1}{2}\big(v(2) - v(1)\big) + \frac{1}{2}\big(v(2, 3) - v(1, 3)\big) > 0$$

On the other hand

$$\phi_3(v) - \phi_2(v) = \frac{1}{2}\big(v(3) - v(2)\big) + \frac{1}{2}\big(v(1, 3) - v(1, 2)\big) > 0.$$

$(\frac{1}{2} = p_0 + p_1 = p_1 + p_2.)$

p-aligned also for other semivalues

Note that $\sqsupseteq^a$ is **p**-aligned with every regular semivalue such that $p_0 \geq p_2$:

$$\pi_2^p(v) - \pi_1^p(v) = (p_0 + p_1)\big(v(2) - v(1)\big) + (p_1 + p_2)\big(v(2,3) - v(1,3)\big) > 0$$

On the other hand

$$\pi_3^p(v) - \pi_2^p(v) = (p_0 + p_1)\big(v(3) - v(2)\big) + (p_1 + p_2)\big(v(1,3) - v(1,2)\big) > 0$$

for every $v \in V(\sqsupseteq^a)$.

Total preorder **p**-aligned for no regular semivalues

It is quite possible that for a given preorder there is no **p**-ordinal regular semivalue associated to it ($(1, 0, \dots, 0)$ does always the job). With $N = \{1, 2, 3\}$ and the following total preorder:

$$N \sqsupset \{1, 2\} \sqsupset \{2, 3\} \sqsupset \{1\} \sqsupset \{1, 3\} \sqsupset \{2\} \sqsupset \{3\} \sqsupset \emptyset.$$

Then 1 and 2 cannot be ordered since, for every fixed semivalue **p** the quantity

$$(p_0 + p_1)(v(\{1\}) - v(\{2\})) + (p_1 + p_2)(v(\{1, 3\}) - v(\{2, 3\}))$$

can be made both positive and negative by suitable choices of v.

A geometric characterization of alignment

Theorem

Given a total order $\sqsupseteq$ on 2^N, the set of regular semivalues $\sqsupseteq$ is aligned with, is either empty or at least two dimensional convex set.

Can be exactly two dimensional also with several players.
Example Let $N = \{1, 2, 3, 4, 5\}$. A trichotomous preorder such

$$VG = \{\{1\}, \{1,3\}, \{2\}, \{2,3,4\}, \{4,5\}, \{1,2,5\}, \{1,2,3,4\}\},$$
$$G = \{\{3\}, \{1,3,5\}, \{4\}, \{1,2,4,5\}, \{2,4\}, \{3,5\}\},$$
$$B = \{2^N \setminus \{VG \cup G\}\}.$$

Such a total preorder is aligned for every regular semivalue of the form

$$\mathbf{p} = (p_0, p_1, \frac{1 - p_0 - 2p_1}{2}, p_1, \frac{1 - p_0 - 2p_1}{2}).$$

Proposition Let $\sqsupseteq$ be a total preorder on 2^N. If $\sqsupseteq$ satisfies the RESP property, then it is p-aligned with every semivalue π^p. ■

- All the extensions from the literature listed in the previous slide are **p**-aligned with all regular semivalues...

Axiom[Permutational Responsiveness, PR]

We denote by Σ_{ij}^s the set of all subsets of N of cardinality s which do not contain neither i nor j, i.e. $\Sigma_{ij}^s = \{S \in 2^N : i, j \notin S, |S| = s\}$.

Order the sets $S_1, S_2, \dots, S_{n_s}$ in Σ_{ij}^s when you add i and j, respectively:

$$\begin{array}{ccc}
S_1 \cup \{i\} & \sqsupseteq & S_{l(1)} \cup \{j\} \\
\mid\sqcup & & \mid\sqcup \\
S_2 \cup \{i\} & \sqsupseteq & S_{l(2)} \cup \{j\} \\
\mid\sqcup & & \mid\sqcup \\
\dots & \sqsupseteq & \dots \\
\mid\sqcup & & \mid\sqcup \\
S_{n_s} \cup \{i\} & \sqsupseteq & S_{l(n_s)} \cup \{j\}
\end{array}$$

$$\Leftrightarrow \{i\} \sqsupseteq \{j\}$$

Again a sufficient condition...

Proposition Let $\sqsupseteq$ be a total preorder on 2^N. If $\sqsupseteq$ satisfies the PR property, then $\sqsupseteq$ is **p**-aligned with every semivalue. ■

- $\{1,2,3,4\} \sqsupset^b \{2,3,4\} \sqsupset^b \{3,4\} \sqsupset^b \{4\} \sqsupset^b \{3\} \sqsupset^b \{2\} \sqsupset^b \{2,4\} \sqsupset^b \{1,4\} \sqsupset^b \{1,3\} \sqsupset^b \{2,3\} \sqsupset^b \{1,3,4\} \sqsupset^b \{1,2,4\} \sqsupset^b \{1,2,3\} \sqsupset^b \{1,2\} \sqsupset^b \{1\} \sqsupset^b \emptyset$ is **p**-aligned for all p but does not satisfy the PR property.

Axiom[Double Permutational Responsiveness, DPR]

Order the sets $S_1, S_2, \dots, S_{n_s+n_{s-1}}$ in $\Sigma_{ij}^{s} \cup \Sigma_{ij}^{s-1}$ when you add i and j, respectively:

$$\begin{array}{ccc} S_1 \cup \{i\} & \sqsupseteq & S_{I(1)} \cup \{j\} \\ |\sqcup & \sqsupseteq & |\sqcup \\ S_2 \cup \{i\} & & S_{I(2)} \cup \{j\} \\ |\sqcup & & |\sqcup \\ \dots & \sqsupseteq & \dots \\ |\sqcup & & |\sqcup \\ S_{n_s+n_{s-1}} \cup \{i\} & \sqsupseteq & S_{I(n_s+n_{s-1})} \cup \{j\} \end{array}$$

$$\Leftrightarrow \{i\} \sqsupseteq \{j\}$$

A characterization with possibility of interaction

Theorem

The following statements are equivalent:

1) $\sqsupseteq$ *fulfills the DPR property;*
2) $\sqsupseteq$ *is* **p***-aligned for all semivalues.*

- $\{1,2,3,4\} \sqsupset^b \{2,3,4\} \sqsupset^b \{3,4\} \sqsupset^b \{4\} \sqsupset^b \{3\} \sqsupset^b \{2\} \sqsupset^b \{2,4\} \sqsupset^b \{1,4\} \sqsupset^b \{1,3\} \sqsupset^b \{2,3\} \sqsupset^b \{1,3,4\} \sqsupset^b \{1,2,4\} \sqsupset^b \{1,2,3\} \sqsupset^b \{1,2\} \sqsupset^b \{1\} \sqsupset^b \emptyset$ is **p**-aligned for all **p**, is not PR, but it is DPR.

Finding semivalues aligned with $\sqsupseteq$

Let $\sqsupseteq$ be a total preorder on 2^N. For each $A \in 2^N$, let $\mathcal{P}^s_{ij}(\sqsupseteq, A)$ be the set of all subsets T containing neither i nor j and with cardinality s such that $T \cup \{i\}$ is weakly preferred to S, i.e.
$\mathcal{P}^s_{ij}(\sqsupseteq, A) = \{S \in \Sigma^s_{ij} : S \cup \{i\} \sqsupseteq A\}$.

Theorem

Let $\sqsupseteq$ be a total preorder on 2^N and consider a semivalue $\mathbf{p} = (p_0, \ldots, p_{n-1})$. Then $\sqsupseteq$ is $\mathbf{p}$-aligned if and only if for all $i, j \in N$ and all $A \in 2^N$

$$\sum_{s=0}^{n-2} (p_s + p_{s+1}) \left(|\mathcal{P}^s_{ij}(\sqsupseteq, A)| - |\mathcal{P}^s_{ji}(\sqsupseteq, A)|\right) \geq 0 \Leftrightarrow \{i\} \sqsupseteq \{j\},$$

Finding semivalues aligned with $\sqsupseteq$ is transformed in a (almost) *classical system of linear inequalities.*

Axiom[Weighted Permutational Responsiveness, WPR]

Let $\mathbf{p}$ be a semivalue with rational coordinates and let $\mathbf{v}$ be a multiple of $\mathbf{p}$ in $\mathbb{N}^n$. Let $x_s = v_s + v_{s+1}$. Order all sets in decreasing order, with repetitions $S_1, S_2, \dots, S_{2^{n-2}}$ in $2^{N\setminus\{i,j\}}$ when you add i and j, respectively:

$$
\begin{array}{lllll}
\text{repeated } x_{s_1} \text{ times} \left\{ \begin{array}{l} S_1 \cup \{i\} \\ \dots \\ S_1 \cup \{i\} \end{array} \right. & \begin{array}{c} \sqsupseteq \\ \dots \\ \sqsupseteq \\ \end{array} & \left. \begin{array}{l} S_{l(1)} \cup \{j\} \\ \dots \\ \dots \\ S_{l(1)} \cup \{j\} \end{array} \right\} \text{repeated } x_{s_{l(1)}} \text{ times} \\
\quad |\sqcup & & |\sqcup \\
\quad \dots & \sqsupseteq & \dots \\
\quad |\sqcup & & |\sqcup \\
\text{repeated } x_{s_{2^{n-2}}} \text{ times} \left\{ \begin{array}{l} S_{2^{n-2}} \cup \{i\} \\ \dots \\ \dots \\ S_{2^{n-2}} \cup \{i\} \end{array} \right. & \begin{array}{c} \sqsupseteq \\ \dots \\ \sqsupseteq \\ \end{array} & \left. \begin{array}{l} S_{l(2^{n-2})} \cup \{j\} \\ \dots \\ S_{l(2^{n-2})} \cup \{j\} \end{array} \right\} \text{repeated } x_{s_{l(2^{n-2})}} \text{ times}
\end{array}
$$

Example

Let $N = \{1, 2, 3\}$ and consider the order

$$N \sqsupset \{1\} \sqsupset \{2,3\} \sqsupset \{1,3\} \sqsupset \{2\} \sqsupset \{1,2\} \sqsupset \{3\} \sqsupset \emptyset.$$

$\mathbf{v} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ $\mathbf{v} = (2, 1, 1)$. Then consider players 1 and 2

$$\begin{array}{ccc} \{1\} & \sqsupset & \{2,3\} \\ \{1\} & \sqsupset & \{2,3\} \\ \{1\} & \sqsupset & \{2\} \\ \{1,3\} & \sqsupset & \{2\} \\ \{1,3\} & \sqsupset & \{2\} \end{array}$$

A simple algorithm to check **p**-alignment

Theorem

Let $\sqsupseteq$ be a total preorder on 2^N and consider a semivalue $\mathbf{p} = (p_0, \ldots, p_{n-1})$, with rational p. Then $\sqsupseteq$ is $\mathbf{p}$-aligned if and only if the property WPR holds.