Concepts of similarities for utility functions

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[1] Joint work with G. Beer, A.Caserta, G. Di Maio

Preferences of the consumers/agents

We are given

1. X, a metric space (*the commodity space*);
2. agents with preferences on subsets of X.

Our (final) goal

To define and study the idea of agents having similar preferences

A preference relation on a set D is a subset $\mathcal{R}$ of $D \times D$ such that:

- $(d, d) \in \mathcal{R}$ for all $d \in D$; (reflexivity)
- $(d, e) \in \mathcal{R}$ and $(e, f) \in \mathcal{R}$ imply $(d, f) \in \mathcal{R}$; (transitivity)
- for $d, e \in D$, either $(d, e) \in \mathcal{R}$ or $(e, d) \in \mathcal{R}$, or both. (completeness)[2]

A basic problem: to associate a utility function to $\mathcal{R}$:

Definition

A utility function for the preference relation is a function $u : D \to \mathbb{R}$ such that:

$$u(d) \geq u(e) \Leftrightarrow d \succeq e.$$

[2] I shall write, as usual, $d \succeq e$ rather than $(d, e) \in \mathcal{R}$

Possible generalizations

- no completeness
- no transitivity (but weaker forms anyway)

This requires adaptations of the definition of utility function:

f.i.

$$d \succeq e \Rightarrow u(d) \geq u(e), d \succ e \Rightarrow u(d) > u(e)^{3}.$$

But also other approaches are considered (f.i. Multi-utility representation: Evren-Ok,NP-preference structures, Giarlotta-Greco)

[3] $d \succ e$ means $(d, e) \in \mathcal{R} \wedge (e, d) \notin \mathcal{R}$.

Further possible generalizations

A utility representation of $(X, \prec)$ is a pair (U, f), where

- $(U, <)$ is the base chain of the representation,
- $f : X \hookrightarrow U$ is a utility function, i.e., an order preserving embedding: $x \prec y \Leftrightarrow f(x) < f(y)$ for each $x, y \in X$.

Classical case, the base chain is $(\mathbb{R}, <)$

Typical example $\mathbb{R}^n_{lex}$

Herden, Metha
Caserta, Giarlotta, Watson

Existence of utility functions

A utility function needs not to exist:

The simplest example is $\mathbb{R}^2$ with the lexicographical order.

However: Call closed a relation $\succeq$ such that the sets:

$$\{d \in D : d \succeq x\}, \qquad \{d \in D : d \preceq x\}$$

are closed for every $x \in D$.

Then the following result holds:

Theorem

Let X be a separable metric space. Suppose $\succeq$ is closed. Then there exists a continuous utility function for $\succeq$.

Classical result by Debreu.

The Kannai topology

In the paper
Continuity properties of the core of a market

Econometrica, 38, 1970

Kannai defines a topology on the set of continuous preferences on $X = \mathbb{R}^l_+$ which turns out to be equivalent to Kuratowski convergence of the graphs

A subbasis for this topology is defined by means of the sets

$$A_{ij} = \{\prec : x \prec y, \forall x \in B_i, \forall y \in B_j\},$$

where B_k is an enumeration of all balls centered at rational points of X and with rational radius.

First example of a topology on consumer's preferences

The space of partial maps

Definition

A partial map between the metric spaces (X, d) and (Y, ρ) is a pair (D, u), where $D \in CL(X)$, $u : D \to Y$ is a map.

The partial map $(\hat{D}, \hat{u})$ is called extension of the map (D, u) if $D \subset \hat{D}$ and $\hat{u}(d) = u(d)$ for all $d \in D$. It is called a restriction of the map (D, u) if $\hat{D} \subset D$ and $\hat{u}(d) = u(d)$ for all $d \in \hat{D}$.

$\mathcal{P}[X, Y]$ is the family of partial maps from X to Y.

Defining a topology on partial maps

Notation. For open set G, compact set K and I interval of the real line, denote by:

$$[G] = \{(D, u) : D \cap G \neq \emptyset\}, \quad [K : I] = \{(D, u) : u(D \cap K) \subset I\}.$$

Definition

The family of sets $[G], [K : I]$ *is a subbase for a topology on* $\mathcal{P}[X, Y]$*, denoted by* τ_B*, and called the* generalized open-compact *topology.*

The Back topology

Theorem

Let X be a locally compact metric space. Then $(\mathcal{P}[X, Y], \tau_B)$ is separable and completely metrizable.

Proposition

The following are equivalent

- $(D_n, u_n) \to (D, u)$ *for the* τ_B *topology;*
- $D_n \to D$ *in Kuratowski sense and for each* $d_n \in D_n$ *such that* $d_n \to d$*, then* $u_n(d_n) \to u(d)$.

Setting for the basic theorem

X is a locally compact metric space. A continuous preference pair is $(D, \succeq)$ with $D \subset X$ and $\succeq$ preference relation on $D \times D$.

$\mathcal{P}(X)$ is the space of preference pairs endowed with the topology of Kuratowski convergence.

$\mathcal{P}[X, \mathbb{R}]$ is the space of partial maps endowed with τ_B.

Definition

A preference pair is locally non satiated if for every $d \in D$ and every ball S around d there is $c \in D \cap S$ such that $c \succ d$.

$\mathcal{P}_{\text{lns}}(X)$ is the subset of the locally non satiated preference pairs.

The Back-Levin theorem

Theorem

There exists a continuous map $I : \mathcal{P}(X) \to \mathcal{P}[X, \mathbb{R}]$ *such that* $I(D, \succeq)$ *is a utility function for* $\succeq$.

Any such map is a homeomorphism between $\mathcal{P}_{\mathrm{lns}}(X)$ *and* $I(\mathcal{P}_{\mathrm{lns}}(X))$.

Remark The Back-Levin theorem applies to preorders that are not necessarily complete

Notation

Definition

A bornology $\mathcal{B}$ on a metric space (X, d) is a family of subsets of X covering X, closed under taking finite unions, and hereditary. When for every $B \in \mathcal{B}$ there is $\delta > 0$ such that $B^\delta \in \mathcal{B}$, the bornology is called stable by small enlargements. If a bornology contains a small ball around each point, it is called local.A base $\mathcal{B}_0$ for a bornology $\mathcal{B}$ is a subfamily of $\mathcal{B}$ cofinal with respect to inclusion.

Important bornologies:

- $\mathcal{F}$: finite subsets of X;
- $P_0(X)$: nonempty subsets of X;
- $\mathcal{B}_d$: nonempty bounded subsets;
- $\mathcal{K}$: nonempty subsets of X with compact closure;
- $\mathcal{B}_{tb}$: nonempty totally bounded subsets;
- $\mathcal{B}_{UC}$: UC-sets of X.

Basic Definitions

Definition

A net (D_γ, u_γ) is $\mathcal{P}^-(\mathcal{B})$-convergent to (D, u) if for every $B \in \mathcal{B}$ and $\epsilon > 0$:

1. $D \cap B \subset D_\gamma^\epsilon$ *eventually;*
2. $u(D \cap B_1) \subset [u_\gamma(D_\gamma \cap B_1^\epsilon)]^\epsilon$ *eventually, for all* $B_1 \subset B$.

Notation $(D, u) \in \mathcal{P}^-(\mathcal{B}) - \lim(D_\gamma, u_\gamma)$.

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First simple facts

- Taking Y as a one point set provides usual definitions of upper and lower convergences in the hyperspace of the space X. $\mathcal{B}$ the bornology of all sets corresponds to convergence in the Hausdorff metric topology, the bornology of bounded sets corresponds to convergence in the bounded Hausdorff topology. When X is locally compact, the bornology of the relatively compact sets provides Kuratowski convergence (Fell topology);
- If $(D, u) \in \mathcal{P}^-(\mathcal{B}) - \lim(D_\gamma, u_\gamma)$, then $(\hat{D}, \hat{u}) \in \mathcal{P}^-(\mathcal{B}) - \lim(D_\gamma, u_\gamma)$ for any restriction $(\hat{D}, \hat{u})$ of (D, u). Dually, if $(D, u) \in \mathcal{P}^+(\mathcal{B}) - \lim(D_\gamma, u_\gamma)$, then $(\hat{D}, \hat{u}) \in \mathcal{P}^+(\mathcal{B}) - \lim(D_\gamma, u_\gamma)$ for any extension $(\hat{D}, \hat{u})$ of (D, u);
- If $(D, u) \in \mathcal{P}^-(\mathcal{B}) - \lim(D_\gamma, u_\gamma)$, then $(D, u) \in \mathcal{P}^-(\hat{\mathcal{B}}) - \lim(D_\gamma, u_\gamma)$ for any bornology $\hat{\mathcal{B}} \subset \mathcal{B}$. The same happens with upper convergence.

Further simple facts

- Even with fixed domains, lower and upper convergence are different;
- A net might have more than one common limit. If the bornology is local, the limits have a common domain. If a net has a $\mathcal{P}(\mathcal{B})$ continuous limit, then the limit is unique;
- Inside the set of the continuous partial maps, $\mathcal{P}^+(\mathcal{F})$ convergence of a net (D, u_γ) to (D, u) is equivalent to pointwise convergence of (D, u_γ) to (D, u) note:common domains;
- $\mathcal{P}^-(\mathcal{F})$ convergence of (D_γ, u_γ) to (D, u) is equivalent to having, for each $d \in D$, existence of $d_\gamma \in D_\gamma$ such that $(d_\gamma, u_\gamma(d_\gamma)) \to (d, u(d))$.

Two preliminary results

Proposition

Given a bornology on Z and endowing $X \times Y$ of the box bornology $(\mathcal{B}, P_0(X))$, $\mathcal{P}(\mathcal{B})$ convergence of (D_γ, u_γ) to (D, u) is equivalent to set bornological convergence of the graphs in $X \times Y$.

Proposition

Convergence is topological when the bornology is stable for small enlargements. The restriction of $\mathcal{P}(\mathcal{B})$ convergence on partial maps with compact domain is topological.

An alternative way to describe the convergences

Proposition

Let $\mathcal{B}$ be a bornology on X. A net (D_γ, u_γ) is $\mathcal{P}^-(\mathcal{B})$- convergent to (D, u) if and only if for every $B \in \mathcal{B}$ and $\epsilon > 0$, the following two conditions hold:

1. $D \cap B \subset D^\epsilon_\gamma$ *eventually;*
2. $\sup_{z \in D \cap B} \inf_{x \in B_d[z,\epsilon]} \rho(u(z), u_\gamma(x)) < \epsilon$ *eventually.*

Proposition

Let $\mathcal{B}$ be a bornology on X. A net (D_γ, u_γ) is $\mathcal{P}^+(\mathcal{B})$- convergent to (D, u) if and only if for every $B \in \mathcal{B}$ and $\epsilon > 0$, the following two conditions hold:

1. $D_\gamma \cap B \subset D^\epsilon$ *eventually;*
2. $\sup_{z \in D \cap B} \inf_{x \in B_d[z,\epsilon]} \rho(u(x), u_\gamma(z)) < \epsilon$ *eventually.*

Uniform Continuity

Definition

Let $\mathcal{B}$ be a bornology on X and $(D, u) \in \mathcal{P}[X, Y]$. (D, u) is uniformly continuous on the bornology $\mathcal{B}$ if for every $B \in \mathcal{B}$ the map

$$u : D \cap B \to Y$$

is uniformly continuous. (D, u) is strongly uniformly continuous on the bornology $\mathcal{B}$ if for every $B \in \mathcal{B}$ for each $\epsilon > 0$ there is $\delta > 0$ such such that if $d(x, w) < \delta$ and $\{x, w\} \cap (B \cap D) \neq \emptyset$, then $\rho(u(x), u(w)) < \epsilon$.

1. (D, u) is strongly uniformly continuous on $\mathcal{K}$ if and only if u is continuous at each point of D;
2. (D, u) is strongly uniformly continuous on $\mathcal{B}$ if and only if it is uniformly continuous on $\mathcal{B}$, provided $\mathcal{B}$ is stable under small enlargements.

Another characterization of upper convergence

Proposition

Let $\mathcal{B}$ be a bornology on X and let (D, u) be strongly uniformly continuous on $\mathcal{B}$. TFAE

1. (D_γ, u_γ) *is* $\mathcal{P}^+(\mathcal{B})$*- convergent to* (D, u)*;*
2. *for every* $B \in \mathcal{B}$ *and* $\epsilon > 0$*:*
   1. $D_\gamma \cap B \subset D^\epsilon$ *eventually;*
   2. $\sup_{z \in D_\gamma \cap B} \sup_{x \in B_d[z,\epsilon]} \rho(u(x), u_\gamma(z)) < \epsilon$ *eventually.*

A further simplification on convergence

Theorem

Let $\mathcal{B}$ be a bornology stable with respect small enlargement and let (D, u) be uniformly continuous on $\mathcal{B}$. TFAE:

1. (D_γ, u_γ) *is* $\mathcal{P}(\mathcal{B})$*- convergent to* (D, u)*;*
2. $\forall \epsilon > 0$ *it holds that:*
    1. $D \cap B \subset D_\gamma^\epsilon$*, eventually;*
    2. $D_\gamma \cap B \subset D^\epsilon$ *eventually;*
    3. $\sup_{z \in D_\gamma \cap B} \sup_{x \in B_d[z,\epsilon]} \rho(u(x), u_\gamma(z)) < \epsilon$ *eventually.*

The same holds for the bornologies of the totally bounded sets and of the relatively compact sets.

A due theorem

Theorem

Let X be a locally compact metric space. Then the topology τ_B coincides with the topology of $\mathcal{P}(\mathcal{K})$ convergence.

The case of fixed domains

Theorem

Suppose (D, u_γ), (D, u) are partial maps (with common domain). Then

- *if (D, u_γ) uniformly converges to (D, u) on the bornology $\mathcal{B}$, then it converges for the topology $\mathcal{P}(\mathcal{B})$;*
- *the converse is true provided (D, u) is strongly uniformly continuous on $\mathcal{B}$.*

Comparing two upper convergences

Remember $\mathcal{P}^+(\mathcal{B})$ convergence: for all $B \in \mathcal{B}$ and $\epsilon > 0$:

1. $D_\gamma \cap B \subset D^\epsilon$ eventually;
2. $u_\gamma(D_\gamma \cap B_1) \subset [u(C \cap B_1^\epsilon)]^\epsilon$ eventually, for all $B_1 \subset B$.

Compare with for all $B \in \mathcal{B}$ and $\epsilon > 0$

1. $D_\gamma \cap B \subset D^\epsilon$ eventually;
2. $u_\gamma(D_\gamma \cap B) \subset [u(C \cap B^\epsilon)]^\epsilon$ eventually.

The second one has the same structure as the definition in K. Back. Call it $\mathcal{A}^+(\mathcal{B})$ convergence.

Are they the same?

NO!

BUT

Theorem

Let X be a metric space, let $\mathcal{B}$ be a bornology. Then on the space of the maps strongly uniformly continuous on bornology a sequence (D_n, u_n) converges for $\mathcal{P}^+(\mathcal{B})$ if and only if it converges for $\mathcal{A}^+(\mathcal{B})$.

Theorem

Let X be a metric space. Then on the space of the continuous maps a net (D_γ, u_γ) converges for $\mathcal{P}^+(\mathcal{B}_{tb})$ if and only if it converges for $\mathcal{A}^+(\mathcal{B}_{tb})$.

Final result

Theorem

Let X be a metric space, $\mathcal{B}$ a bornology on X and suppose Y is totally bounded. Then on $\mathcal{P}[X, Y]$ a net (D_γ, u_γ) converges for $\mathcal{P}^+(\mathcal{B})$ if and only if it converges for $\mathcal{A}^+(\mathcal{B})$.

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